

Crossed simplicial groups and structured surfaces

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Dedicated to the memory of Jean-Louis Loday

Abstract

We propose a generalization of the concept of a Ribbon graph suitable to provide combinatorial models for marked surfaces equipped with a G -structure. Our main insight is that the necessary combinatorics is neatly captured in the concept of a crossed simplicial group as introduced, independently, by Krasauskas and Fiedorowicz-Loday. In this context, Connes' cyclic category leads to Ribbon graphs while other crossed simplicial groups naturally yield different notions of structured graphs which model unoriented, N -spin, framed, etc, surfaces. Our main result is that structured graphs provide orbicell decompositions of the respective G -structured moduli spaces. As an application, we show how, building on our theory of 2-Segal spaces, the resulting theory can be used to construct categorified state sum invariants of G -structured surfaces.

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Introduction

Ribbon graphs form a fundamental tool in the combinatorial study of moduli spaces of Riemann surfaces and of the associated mapping class groups [42, 57]. Similarly, they appear in string theory and in perturbative expansions of matrix integrals [49].

The first goal of this paper is to propose a generalization of ribbon graphs which governs, in an analogous way, the geometry and topology of *structured surfaces*. By these we mean C^∞ -surfaces S , possibly with boundary, equipped with a nonempty set $M \subset S$ of marked points together with a reduction of the structure group of the tangent bundle $T_{S \setminus M}$ along a fixed Lie group homomorphism

$$(0.1) \quad \mathfrak{p} : \mathbb{G} \longrightarrow GL(2, \mathbb{R}).$$

We assume that \mathfrak{p} is a *connective covering*: a not necessarily surjective unramified covering such that the preimage of the component of identity $GL^+(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ is connected. This implies that a \mathbb{G} -structure on S is a discrete datum. A $GL^+(2, \mathbb{R})$ -structure is an orientation; in the case when \mathbb{G} is an N -fold covering of $GL^+(2, \mathbb{R})$, a \mathbb{G} -structure is known as an *N -spin structure*, etc. Fixing a topological type of a \mathbb{G} -structured marked surface (S, M) , we then have the structured mapping class group $\text{Mod}^{\mathbb{G}}(S, M)$.

For a more analytic point of view, consider the subgroup of conformal linear transformations

$$\text{Conf}(2) = (\mathbb{Z}/2) \ltimes \mathbb{C}^* \subset GL(2, \mathbb{R})$$

which is homotopy equivalent to $GL(2, \mathbb{R})$, so that connective coverings of the two groups are in bijection. Denoting by G_{conf} the preimage of $\text{Conf}(2)$ in \mathbb{G} , we can consider surfaces with G_{conf} -structure which are essentially algebro-geometric objects: if \mathbb{G} preserves orientation, we obtain Riemann surfaces, otherwise Klein surfaces [2]. Such objects have moduli spaces \mathcal{M}^G (more precisely, stacks) which are algebro-geometric counterparts of the groups $\text{Mod}^{\mathbb{G}}(S)$.

These groups and moduli spaces are best known for the “standard” case $\mathbb{G} = GL^+(2, \mathbb{R})$ (oriented surfaces, Riemann surfaces), see, e.g., [24]. Other cases are attracting increasingly more interest in recent years. For example, in the unoriented case $\mathbb{G} = GL(2, \mathbb{R})$, the orbifold \mathcal{M}^G is well known to be the real locus of the moduli stack of algebraic curves [54, 61, 44]. The unoriented mapping class groups, although classical [50, 62, 7], have some of their important properties established only recently [58, 64]. The situation is similar for N -spin mapping class groups [59]; the corresponding moduli spaces \mathcal{M}^G of N -spin Riemann surfaces [35] provide important examples of integrable hierarchies and cohomological field theories [36].

All this makes it desirable to have a flexible combinatorial formalism extending that of ribbon graphs to the case of an arbitrary \mathbb{G} as above. Our main observation is that the ingredients of such a formalism can be found in the concept of a *crossed simplicial group* (due to Fiedorowicz-Loday [25] and Krasauskas [43]). A crossed simplicial group is

a certain category $\Delta\mathfrak{G}$ with objects $[n]$, $n \geq 0$, containing the simplicial category Δ . It turns out that each connective covering \mathbb{G} as in (0.1) has its associated crossed simplicial group $\Delta\mathfrak{G}$. The prime example is the *cyclic category* Λ introduced by A. Connes [13] as the foundation of cyclic homology. For this category $\mathfrak{G}_n = \text{Aut}([n]) = \mathbb{Z}/(n+1)$ is the cyclic group. This matches the data of a cyclic ordering on the set of halfedges incident to a vertex of a ribbon graph. Thus Λ can be said to “govern” the world of oriented surfaces.

More generally, for each \mathbb{G} as above, with associated crossed simplicial group $\Delta\mathfrak{G}$, we introduce the concept of a $\Delta\mathfrak{G}$ -structured graph. We show that any embedding of a graph Γ into a \mathbb{G} -structured surface induces a $\Delta\mathfrak{G}$ -structure on Γ (Proposition 4.1.8). We further prove (Theorem 4.2.3) that the nerve of the category formed by $\Delta\mathfrak{G}$ -structured graphs and their contractions, is homotopy equivalent to the union of the classifying spaces of the groups $\text{Mod}^{\mathbb{G}}(S, M)$ for all topological types of stable marked surfaces (S, M) .

For example, for unoriented surfaces, the cyclic category Λ is replaced by the *dihedral category* Ξ , see [46]. Applying our formalism to Ξ , we get a concept known as a *Möbius graph* [10, 52, 53] but formulated in a somewhat more conceptual way.

The same way Ribbon graphs can be utilized to construct invariants of oriented surfaces, $\Delta\mathfrak{G}$ -structured graphs provide means to construct invariants of \mathbb{G} -structured surfaces. While this includes generalizations of 2-dimensional oriented topological field theories constructed from Frobenius algebras, we are mainly interested in a “categorified” variant of this construction: A functor $X : \Delta\mathfrak{G}^{\text{op}} \rightarrow \mathcal{C}$ can be seen as a simplicial object X in \mathcal{C} together with extra structure given by an action of $\mathfrak{G}_n = \text{Aut}([n])$ on X_n for every n . We can evaluate such X on any $\Delta\mathfrak{G}$ -structured graph Γ to obtain an object $X(\Gamma)$ in \mathcal{C} . Assuming that X satisfies a certain combinatorial descent condition (2-Segal condition) introduced in [22], we can think of $X(\Gamma)$ as the global sections of a combinatorial sheaf on the surface (S, M) modelled by Γ , so that $X(\Gamma) \cong X(S, M)$ is independent on the chosen graph Γ . Further, if \mathcal{C} carries a model structure, then we have a derived variant of this construction which generalizes the invariants of [23] obtained from 2-Segal cyclic objects.

For example, a 2-Segal dihedral object X associates to every stable marked unoriented surface (S, M) , an object $X(S, M)$ with a coherent action of the unoriented mapping class group of (S, M) .

A cyclic 2-Segal object X in \mathcal{C} can be seen as a nonlinear, categorical analog of a Frobenius algebra A . If $\mathcal{C} = \text{Set}$, then 1-simplices, i.e., elements of X_1 are analogous to elements of A , and the number of 2-simplices $\sigma \in X_2$ with three given boundary 1-simplices a, b, c corresponds to the cyclically invariant scalar product (ab, c) . The construction of [23] is thus a “categorification” of the celebrated fact that Frobenius algebras (and, more general, Calabi-Yau dg-categories) give rise to invariants of oriented marked surfaces. In other words, it fits into Table 1 summarizing various types of “invariants” and their meaning.

It is natural therefore to expect that our approach can be developed to include “structured” analogs of all the other rows in Table 1. In this paper we discuss only the structured analog of the concept of a Frobenius algebra, leaving other contexts for future work. This

Source of invariants	Type of invariants	Precise meaning
Usual Frobenius algebras	Numerical invariants	Elements of H^0 of moduli spaces
Calabi-Yau algebras and categories	Cohomological invariants	Higher cohomology classes on moduli spaces [42][16]
Modular categories	Vector space-valued invariants (fusion data)	Local systems of vector spaces on moduli spaces [6]
Cyclic 2-Segal objects	Categorical invariants	Local systems of objects of \mathcal{C} on moduli spaces

Table 1: Various “invariants” of oriented surfaces.

analog is based on the following concept.

Let H be a group equipped with a *parity*, by which we mean a homomorphism

$$\rho : H \longrightarrow \mathbb{Z}/2.$$

Let us write H_0 and H_1 for the preimages of 0 and 1 whose elements are called even and odd, respectively. One can then introduce the concept of a *twisted action* of H on an associative algebra so that even elements of H act by automorphisms while odd elements of H act by anti-automorphisms. One can similarly speak about twisted actions of H on a category: even group elements of act by covariant functors while odd elements act by contravariant functors. In the situation of a group \mathbb{G} and a crossed simplicial group $\Delta\mathfrak{G}$ as above, the group \mathfrak{G}_0 comes with a natural parity. It turns out that considering algebras and categories with twisted \mathfrak{G}_0 -action allows us to extend many known classical results to the \mathbb{G} -structured situation. In particular:

- (1) For a category \mathcal{C} with a twisted \mathfrak{G}_0 -action we have a natural “nerve” $N^{\mathbb{G}}(\mathcal{C})$ which is a $\Delta\mathfrak{G}$ -set.
- (2) For an algebra A with a twisted \mathfrak{G}_0 -action, the Hochschild complex $C_{\bullet}^{\text{Hoch}}(A)$ has a natural structure of a $\Delta\mathfrak{G}$ -vector space.

We expect that a Frobenius algebra (resp. a Calabi-Yau category) with a twisted \mathfrak{G}_0 -action gives rise to a numerical (resp. cohomological, in the sense of Table 1) invariant of \mathbb{G} -structured marked surfaces. The results of [1, 56] for $\Delta\mathfrak{G} = \Xi, \Lambda_2$ (unoriented surfaces, 2-spin surfaces) support this expectation.

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1 Crossed simplicial groups and planar Lie groups

1.1 Basic definitions

At the basis of combinatorial topology lies the *simplex category* Δ whose objects are the finite ordinals $[n] = \{0, 1, \dots, n\}$, $n \geq 0$, with morphisms given by monotone maps. *Simplicial objects* in a category \mathbf{C} are functors $\Delta^{\text{op}} \rightarrow \mathbf{C}$. In his axiomatization of cyclic homology, A. Connes introduced a category Λ which can be thought of as a hybrid of Δ and the family $\{\mathbb{Z}/(n+1)\}$ of finite cyclic groups which appear as the automorphism groups of the objects of Λ , see [13, 46]. The formal interplay between the two is captured in the notion of a crossed simplicial group introduced in [25, 43].

Definition 1.1.1. A *crossed simplicial group* is a category $\Delta\mathfrak{G}$ equipped with an embedding $i : \Delta \rightarrow \Delta\mathfrak{G}$ such that:

- (1) the functor i is bijective on objects,
- (2) any morphism $u : i[m] \rightarrow i[n]$ in $\Delta\mathfrak{G}$ can be uniquely expressed as a composition $i(\phi) \circ g$ where $\phi : [m] \rightarrow [n]$ is a morphism in Δ and g is an automorphism of $i[m]$ in $\Delta\mathfrak{G}$.

We will refer to the representation $u = i(\phi) \circ g$ in (2) as the *canonical factorization* of u . To keep the notation light, we will usually leave the embedding i implicit, referring to the objects of $\Delta\mathfrak{G}$ as $[n]$, $n \geq 0$. To every crossed simplicial group $\Delta\mathfrak{G}$, we can associate a sequence of groups

$$(1.1.2) \quad \mathfrak{G}_n = \text{Aut}_{\Delta\mathfrak{G}}([n]).$$

Further, by Property (2), any diagram

$$\begin{array}{ccc} & & [n] \\ & & \downarrow g \\ [m] & \xrightarrow{\phi} & [n] \end{array}$$

where ϕ in Δ and $g \in \mathfrak{G}_n$, can be uniquely completed to a commutative diagram in $\Delta\mathfrak{G}$

$$(1.1.3) \quad \begin{array}{ccc} [m] & \xrightarrow{g^*\phi} & [n] \\ \phi^*g \downarrow & & \downarrow g \\ [m] & \xrightarrow{\phi} & [n] \end{array}$$

with $g^*\phi$ in Δ and $\phi^*g \in \mathfrak{G}_n$. These data satisfy the following compatibilities.

Proposition 1.1.4. (a) For every morphism $\phi : [m] \rightarrow [n]$ in Δ , the correspondence $g \mapsto \phi^* g$ defines a map of sets

$$\phi^* : \mathfrak{G}_n \longrightarrow \mathfrak{G}_m, \quad g \mapsto \phi^* g$$

The association $\phi \mapsto \phi^*$ makes the family $\mathfrak{G} = (\mathfrak{G}_n)_{n \geq 0}$ into a simplicial set. These maps preserve unit elements but not necessarily the group structure.

(b) For objects $[m], [n]$, the association

$$(g, \phi) \mapsto g^* \phi$$

determines a right action of \mathfrak{G}_n on the set $\text{Hom}_\Delta([m], [n])$. In the case $m = n$, this action preserves the identity morphism.

(c) In addition, we have the identities

$$\begin{aligned} \phi^*(g \circ h) &= \phi^* g \circ (g^* \phi)^* h, \\ g^*(\phi \circ \psi) &= g^* \phi \circ (\phi^* g)^* \psi. \end{aligned}$$

(d) Conversely, any sequence of groups $\mathfrak{G} = (\mathfrak{G}_n)$ with operations

$$(g, \phi) \mapsto (\phi^*(g), g^*(\phi))$$

satisfying the compatibilities (a) - (c), uniquely determines a crossed simplicial group.

Proof. [25], Proposition 1.6. □

Example 1.1.5. Note that a simplicial group is a particular example of a crossed simplicial group, corresponding to trivial actions of \mathfrak{G}_n on $\text{Hom}_\Delta([m], [n])$. Therefore, in general, these actions and the identities in Proposition 1.1.4(c), describe the “deviation” of \mathfrak{G} from being a simplicial group.

Example 1.1.6. Let us point out the following extreme cases of canonical factorization of morphisms in $\Delta\mathfrak{G}$:

$$\text{Hom}_{\Delta\mathfrak{G}}([n], [0]) \cong \mathfrak{G}_n, \quad \text{Hom}_{\Delta\mathfrak{G}}([0], [n]) \cong \{0, 1, \dots, n\} \times \mathfrak{G}_0.$$

The first identification follows from the fact that $\text{Hom}_\Delta([n], [0]) = \text{pt}$. Note that the simplicial set structure on $\{\mathfrak{G}_n\}$ can be deduced immediately from this identification: $\text{Hom}_{\Delta\mathfrak{G}}(-, [0])$ is a contravariant functor on $\Delta\mathfrak{G}$ and, by restriction, on Δ .

The second identification follows by realizing $\{0, 1, \dots, n\}$ as $\text{Hom}_\Delta([0], [n])$. It can be interpreted by saying that the sets $\text{Hom}_{\Delta\mathfrak{G}}([0], [n])$ “grow linearly with n ”.

The following proposition means that objects of $\Delta\mathfrak{G}$ can be considered as sets with extra structure, a point of view we elaborate in Chapter 2.

Proposition 1.1.7. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group. Then we have a functor*

$$\lambda : \Delta\mathfrak{G} \longrightarrow \mathcal{Set}, [n] \mapsto \text{Hom}_{\Delta\mathfrak{G}}([0], [n]) / \mathfrak{G}_0$$

where $\lambda([n])$ can be canonically identified with the set $\text{Hom}_{\Delta}([0], [n]) \cong \{0, 1, \dots, n\}$. In particular, we obtain, for every object $[n]$ of $\Delta\mathfrak{G}$, a canonical group homomorphism

$$\lambda_n : \mathfrak{G}_n \longrightarrow S_{n+1}.$$

Proof. Follows from the second identification in Example 1.1.6. \square

1.2 The Weyl crossed simplicial group

Let J be a finite set. A *signed linear order* on J consists of

- (1) a linear order on J ,
- (2) a map of sets $\varepsilon : J \rightarrow \mathbb{Z}/2\mathbb{Z}$.

We introduce a category $\Delta\mathfrak{W}$ with objects given by the sets $\{0, 1, \dots, n\}$, $n \geq 0$. A morphism $f : I \rightarrow J$ is given by a map of underlying sets together with the choice of a signed linear order on each fiber $f^{-1}(j)$. Composition of morphisms is obtained by forming *lexicographic signed linear orders*: Given $f : I \rightarrow J$ and $g : J \rightarrow K$, we have, for $k \in K$, a linear order on $(gf)^{-1}(k)$ obtained by declaring $i_1 \leq i_2$ if

- (1) either $f(i_1) = f(i_2)$ and $i_1 \leq i_2$ with respect to the linear order on the fiber of f ,
- (2) or $f(i_1) \neq f(i_2)$ and $f(i_1) \leq f(i_2)$ with respect to the linear order on the fiber of g .

The sign of $i \in (gf)^{-1}(k)$ is obtained by setting $\varepsilon_{gf}(i) = \varepsilon_f(i) + \varepsilon_g(f(i))$.

Proposition 1.2.1. *The category $\Delta\mathfrak{W}$ is a crossed simplicial group with*

$$\mathfrak{G}_n = \text{Aut}_{\Delta\mathfrak{W}}([n]) \cong W_{n+1}$$

where W_{n+1} denotes the signed permutation group of $\{0, 1, \dots, n\}$ also known as the wreath product $\mathbb{Z}/2\mathbb{Z} \wr S_{n+1}$.

Proof. We have to verify the unique factorization property which is a direct consequence of the definition of $\Delta\mathfrak{W}$. \square

Following Krasauskas, we call $\Delta\mathfrak{W}$ the *Weyl crossed simplicial group*, since the group W_n is the Weyl group of the root system B_n (or C_n). Its fundamental importance stems from the following result ([43, 25]).

Theorem 1.2.2. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group.*

- (1) *There is a canonical functor $\pi : \Delta\mathfrak{G} \rightarrow \Delta\mathfrak{W}$.*

(2) For every $n \geq 0$, we have an induced short exact sequence of groups

$$1 \longrightarrow \mathfrak{G}'_n \longrightarrow \mathfrak{G}_n \longrightarrow \mathfrak{G}''_n \longrightarrow 1$$

where \mathfrak{G}'_n and \mathfrak{G}''_n denote kernel and image, respectively, of the induced homomorphism $\pi_n : \mathfrak{G}_n \rightarrow W_{n+1}$. The short exact sequences assemble to a sequence of functors

$$\Delta \mathfrak{G}' \longrightarrow \Delta \mathfrak{G} \longrightarrow \Delta \mathfrak{G}''$$

where $\Delta \mathfrak{G}'$ is a simplicial group (Example 1.1.5) and $\Delta \mathfrak{G}'' \subset \Delta \mathfrak{W}$ is a crossed simplicial subgroup of $\Delta \mathfrak{W}$.

Proof. Consider the functor

$$\lambda : \Delta \mathfrak{G} \longrightarrow \mathcal{S}et, [n] \mapsto \text{Hom}_{\Delta \mathfrak{G}}([0], [n]) / \mathfrak{G}_0$$

from Proposition 1.1.7. We claim that λ admits a canonical factorization over the forgetful functor $\Delta \mathfrak{W} \rightarrow \mathcal{S}et$. Note that, via λ , the group \mathfrak{G}_n acts canonically on the set $\text{Hom}_{\Delta}([0], [n]) = \{0, 1, \dots, n\}$ of vertices of the combinatorial n -simplex. A vertex $i \in \{0, 1, \dots, n\}$ can be canonically identified with the corresponding degeneracy map

$$s_i : [n+1] \longrightarrow [n], j \mapsto \begin{cases} j & \text{for } j \leq i, \\ j-1 & \text{for } j > i. \end{cases}$$

Explicitly, given $g \in \mathfrak{G}_n$ and $i \in \{0, 1, \dots, n\}$, we have a commutative square

$$\begin{array}{ccc} [n+1] & \xrightarrow{s_i} & [n] \\ s_j^* g \downarrow & & \downarrow g \\ [n+1] & \xrightarrow{s_j} & [n] \end{array}$$

where $j = \lambda(g)(i)$. Clearly, the map $\lambda(s_j^* g)$ induces a map from the fiber $s_i^{-1}(i) = \{i, i+1\}$ to the fiber $s_j^{-1}(j) = \{j, j+1\}$ which is either order preserving or order reversing. Therefore, we can lift $\lambda(g) \in S_{n+1}$ to a signed permutation

$$\tilde{\lambda}(g) = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n; \lambda(g)) \in \mathbb{Z}/2\mathbb{Z} \wr S_{n+1}$$

where

$$\varepsilon_i = \begin{cases} 0 & \text{if } \lambda(s_j^* g) : \{i, i+1\} \rightarrow \{j, j+1\} \text{ is order preserving,} \\ 1 & \text{if } \lambda(s_j^* g) : \{i, i+1\} \rightarrow \{j, j+1\} \text{ is order reversing.} \end{cases}$$

Using the unique factorization property in $\Delta \mathfrak{G}$, it is straightforward to verify that the association $g \mapsto \tilde{\lambda}(g)$ extends to provide a functor

$$\tilde{\lambda} : \Delta \mathfrak{G} \longrightarrow \Delta \mathfrak{W}$$

commuting with the forgetful functors to $\mathcal{S}et$ which proves (1). The remaining statements are easy to verify, the only nonobvious point being that $\Delta\mathfrak{G}'$ is in fact noncrossed simplicial. But this follows from the construction of $\tilde{\lambda}$: For $g \in \mathfrak{G}_n$, the condition $\lambda(g) = \text{id} \in S_{n+1}$ implies that g^* fixes all face maps $[n-1] \rightarrow [n]$. The condition that all signs ε_i of the lift $\widetilde{\lambda(g)}$ are 0 implies that g^* fixes all degeneracy maps $[n+1] \rightarrow [n]$. But this implies that $\Delta\mathfrak{G}''$ is a simplicial group (cf. Example 1.1.5). \square

According to Theorem 1.2.2, the classification of crossed simplicial groups up to extensions by simplicial groups therefore reduces to the classification of crossed simplicial subgroups of the Weyl crossed simplicial group $\Delta\mathfrak{W}$. There are precisely 7 such subgroups called *fundamental crossed simplicial groups*. We provide a list in Table 2, following the terminology introduced in [43]. Therefore, every crossed simplicial group has a *type* given by its image in $\Delta\mathfrak{W}$. For example a crossed simplicial group of *trivial* type is a (non-crossed) simplicial group. Note that the 7 subgroups of $\Delta\mathfrak{W}$ can be further distinguished according to their *growth rate*:

- (1) *Constant*: For the trivial and reflexive groups, the cardinality of \mathfrak{G}_n is constant.
- (2) *Tame*: For the cyclic and dihedral groups, the size of \mathfrak{G}_n grows linearly with n .
- (3) *Wild*: For the remaining subgroups, the size of \mathfrak{G}_n grows exponentially.

Type	Subgroup of $\Delta\mathfrak{W}$
Trivial	$\{1\}$
Reflexive	$\{\mathbb{Z}/2\mathbb{Z}\}$
Cyclic	$\{\mathbb{Z}/(n+1)\mathbb{Z}\}$
Dihedral	$\{D_{n+1}\}$
Symmetric	$\{S_{n+1}\}$
Reflexosymmetric	$\{\mathbb{Z}/2\mathbb{Z} \ltimes S_{n+1}\}$
Weyl	$\{\mathbb{Z}/2\mathbb{Z} \wr S_{n+1}\}$

Table 2: The 7 types of crossed simplicial groups.

1.3 Semiconstant crossed simplicial groups and twisted group actions

In this section, we will study a particular class of crossed simplicial groups which are of reflexive type in the sense of Table 2.

Let $\omega_n : [n] \rightarrow [0]$ be the unique morphism from $[n]$ to $[0]$ in Δ . Let $\Delta\mathfrak{G}$ be a crossed simplicial group. The first equality in Proposition 1.1.4(c) together with the uniqueness of ω_n implies that the pullback map

$$\omega_n^* : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_n$$

is a group homomorphism.

Definition 1.3.1. A crossed simplicial group $\Delta\mathfrak{G}$ is called *semiconstant* if, for every $n \geq 0$, the homomorphism $\omega_n^* : \mathfrak{G}_0 \longrightarrow \mathfrak{G}_n$, is an isomorphism.

Note that this condition implies that, for every map $\phi : [m] \rightarrow [n]$ in Δ , the corresponding map $\phi^* : \mathfrak{G}_n \rightarrow \mathfrak{G}_m$ is a group isomorphism. We may use ω_n^* to identify \mathfrak{G}_n with \mathfrak{G}_0 . Via this identification we have, for every $g \in \mathfrak{G}_0$ and every morphism $\phi : [m] \rightarrow [n]$ in Δ , a commutative square

$$(1.3.2) \quad \begin{array}{ccc} [m] & \xrightarrow{g^*\phi} & [n] \\ \phi^*g=g \downarrow & & \downarrow g \\ [m] & \xrightarrow{\phi} & [n] \end{array}$$

in $\Delta\mathfrak{G}$. Therefore, the simplicial set \mathfrak{G}_\bullet corresponding to a semiconstant crossed simplicial group is a constant simplicial group. However, the action $\phi \mapsto g^*\phi$ may be nontrivial which is an additional datum.

Example 1.3.3. Any crossed simplicial group $\Delta\mathfrak{G}$ contains the semiconstant crossed simplicial group $\Delta\{\mathfrak{G}_0\}$ generated by \mathfrak{G}_0 : we restrict ourselves to those automorphisms of $[n]$ which are pullbacks of automorphisms of $[0]$ along ω_n .

The goal of this section is to relate semiconstant crossed simplicial groups to twisted group actions. We will start by introducing some terminology. A (strict) action of a group G on a small category \mathcal{C} , is defined to be a homomorphism $G \rightarrow \text{Aut}_{\text{cat}}(\mathcal{C})$ where Cat denotes the category of small categories.

Definition 1.3.4. Let \mathcal{C} be a category equipped with an action of a group G . We define a category $G \ltimes \mathcal{C}$ called the *semidirect product* of G and \mathcal{C} . The objects of $G \ltimes \mathcal{C}$ are the objects of \mathcal{C} , a morphism from x to y is given by a pair (g, ϕ) where $g \in G$ and $\phi : g.x \rightarrow y$ is a morphism in \mathcal{C} . The composition of morphisms $(g, \phi) : x \rightarrow y$ and $(g', \phi') : y \rightarrow z$ is the morphism $(g'g, \phi' \circ g'.\phi)$

Remark 1.3.5. The action of G on \mathcal{C} can be interpreted as a functor $\mathcal{B}G \rightarrow \text{Cat}$ where $\mathcal{B}G$ denotes the groupoid with one object and automorphism group G . In this context, the semidirect product $G \ltimes \mathcal{C}$ equipped with its natural functor to $\mathcal{B}G$ is known as the Grothendieck construction.

Proposition 1.3.6. *Let G be a group acting on Δ . Then the corresponding semidirect product $G \ltimes \Delta$ is a semiconstant crossed simplicial group. Vice versa, any semiconstant crossed simplicial group is isomorphic to a semidirect product $\mathfrak{G}_0 \ltimes \Delta$.*

Proof. The semidirect product $G \ltimes \Delta$ has the unique factorization property: $(g, \phi) = (1, \phi) \circ (g, \text{id})$ making it a semiconstant crossed simplicial group. Given a semiconstant crossed simplicial group $\Delta\mathfrak{G}$, it follows from (1.3.2) that \mathfrak{G}_0 acts on Δ . It immediately follows from the definition that we may identify $\Delta\mathfrak{G}$ with $\mathfrak{G}_0 \ltimes \Delta$. \square

There is an involution τ on Δ which is the identity on objects and maps a morphism $\phi : [m] \rightarrow [n]$ to the *opposite morphism* $\phi^{\text{op}} : [m] \rightarrow [n]$ defined by $\phi(i) = n+1 - \phi(m+1-i)$. If we replace Δ by the equivalent larger category $\mathbf{\Delta}$ of all nonempty finite ordinals, then τ can be defined more naturally as the functor $\mathbf{\Delta} \rightarrow \mathbf{\Delta}$ sending each ordinal (I, \leq) to the opposite ordinal (I, \geq) .

Proposition 1.3.7. *We have $\text{Aut}_{\text{cat}}(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$ with generator given by τ . In particular, any action of a group G on Δ factors via a homomorphism $G \rightarrow \mathbb{Z}/2\mathbb{Z}$ over the action of $\mathbb{Z}/2\mathbb{Z}$ on Δ given by τ .*

Proof. Well known. □

Remark 1.3.8. As a corollary, we obtain that any semidirect product $G \ltimes \Delta$ admits a canonical functor to $\mathbb{Z}/2\mathbb{Z} \ltimes \Delta$. Interpreting this in the general context of crossed simplicial groups says that for a semiconstant crossed simplicial group $\Delta\mathfrak{G}$, the canonical functor into the Weyl crossed simplicial group of Theorem 1.2.2 factors through the semiconstant crossed simplicial group $\Delta\{\mathbb{Z}/2\mathbb{Z}\}$. Thus, in terms of the classification of crossed simplicial groups, semiconstant crossed simplicial groups are extensions of $\Delta\{\mathbb{Z}/2\mathbb{Z}\}$ by constant simplicial groups.

Let \mathcal{C} be a category and let $\mathcal{B}\mathbb{Z}/2\mathbb{Z}$ denote the groupoid with one object and automorphism group $\mathbb{Z}/2\mathbb{Z}$. We define a *parity* on \mathcal{C} to be a functor $\mathcal{C} \rightarrow \mathcal{B}\mathbb{Z}/2\mathbb{Z}$. Explicitly, we are given a partition $\text{Hom}(x, y) = \text{Hom}(x, y)_0 \sqcup \text{Hom}(x, y)_1$ of every morphism set into *even* and *odd* morphisms such that the composite of morphisms of the same parity is even while the composite of morphisms of opposite parity is odd. Categories with parity naturally form a category where morphisms are given by parity preserving functors, i.e., commutative diagrams of

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad} & \mathcal{D} \\ & \searrow & \swarrow \\ & \mathcal{B}\mathbb{Z}/2\mathbb{Z} & \end{array}$$

Examples 1.3.9. (1) A *parity* on a group G is defined to be a homomorphism $G \rightarrow \mathbb{Z}/2\mathbb{Z}$. We obtain a category with parity by passing to groupoids $\mathcal{B}G \rightarrow \mathcal{B}\mathbb{Z}/2\mathbb{Z}$.

(2) Let \mathcal{D} be a category equipped with an involutive functor $\tau : \mathcal{D} \rightarrow \mathcal{D}$. We may interpret τ as a functor $\mathcal{B}\mathbb{Z}/2\mathbb{Z} \rightarrow \text{Cat}$ and apply the Grothendieck construction to obtain a category with parity $\mathcal{D}_\tau \rightarrow \mathcal{B}\mathbb{Z}/2\mathbb{Z}$. Explicitly, \mathcal{D}_τ has the same objects as \mathcal{D} and a morphism between x and y is given by a pair (ε, f) where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $f : \tau^\varepsilon(x) \rightarrow y$ is a morphism in \mathcal{D} . The category \mathcal{D}_τ is also known as a *semidirect product* denoted $\mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{D}$.

Examples 1.3.10. We give some examples of categories with parities arising as semidirect products with $\mathbb{Z}/2\mathbb{Z}$.

- (1) Consider the category $\mathcal{C}at$ of small categories. The involution $\tau : \mathcal{C} \mapsto \mathcal{C}^{op}$ gives rise to a semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{C}at$. We can consider enriched variants of this construction. For example, given a field \mathbf{k} , we obtain a semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{C}at_{\mathbf{k}}$ where $\mathcal{C}at_{\mathbf{k}}$ denotes the category of small \mathbf{k} -linear categories (categories enriched over $\mathbf{Vect}_{\mathbf{k}}$).
- (2) Consider the simplex category Δ with involution \neg . We obtain a corresponding semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes \Delta$.
- (3) Let \mathcal{C} be any category. The involution \neg of Δ induces an involution on the category \mathcal{C}_{Δ} of simplicial objects in \mathcal{C} . We obtain a semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{C}_{\Delta}$.

We have an adjunction

$$FC : \mathcal{S}et_{\Delta} \longleftrightarrow \mathcal{C}at : N$$

where FC associates to a simplicial set K the free category $FC(K)$ generated by K and N takes a small category \mathcal{C} to its nerve $N(\mathcal{C})$. Note that this adjunction is compatible with the involutions \neg and τ so that we have the following consequence.

Proposition 1.3.11. *We have an adjunction*

$$\begin{array}{ccc} FC : \mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{S}et_{\Delta} & \longleftrightarrow & \mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{C}at : N \\ & \searrow & \swarrow \\ & \mathcal{B}\mathbb{Z}/2\mathbb{Z} & \end{array}$$

of categories with parity.

Given a group G and a category \mathcal{D} , both equipped with parity, we define a *twisted action* of G on an object x of \mathcal{D} to be a parity preserving homomorphism $G \rightarrow \text{Aut}_{\mathcal{D}}(x, x)$ or, in other words, a functor

$$\begin{array}{ccc} \mathcal{B}G & \longrightarrow & \mathcal{D} \\ & \searrow & \swarrow \\ & \mathcal{B}\mathbb{Z}/2\mathbb{Z} & \end{array}$$

over $\mathcal{B}\mathbb{Z}/2\mathbb{Z}$.

Examples 1.3.12. (1) From Example 1.3.10(1), we obtain the concept of a twisted group actions on a (\mathbf{k} -linear) category. Explicitly, this means that even group elements act as covariant functors while odd group elements act via contravariant functors. As a special case, given by a \mathbf{k} -linear category with one object, we obtain the concept of a twisted group action on an associative \mathbf{k} -algebra.

- (2) Example 1.3.10(3) provides us with the concept of a twisted group action on a simplicial set (or more generally simplicial object). Here, even group elements act in an the usual orientation preserving way: all face and degeneracy relations are respected. Odd group elements act in an orientation reversing way so that all face and degeneracy relations are reversed according to the involution $\neg : \Delta \rightarrow \Delta$.

Proposition 1.3.13. *Let $\Delta\mathfrak{G}$ be a semiconstant crossed simplicial group. Then, for any category \mathcal{C} , we have a natural equivalence of categories*

$$\mathcal{C}_{\Delta\mathfrak{G}} \xrightarrow{\cong} \mathfrak{G}_0 - \mathcal{C}_{\Delta}, \quad K \mapsto K|_{\Delta}$$

where

$$\mathfrak{G}_0 - \mathcal{C}_{\Delta} = \text{Fun}_{\mathcal{B}\mathbb{Z}/2\mathbb{Z}}(\mathcal{B}\mathfrak{G}_0, \mathbb{Z}/2\mathbb{Z} \ltimes \mathcal{C}_{\Delta})$$

denotes the category of simplicial objects in \mathcal{C} equipped with a twisted group action of $\mathfrak{G}_0 \rightarrow \text{Aut}(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. This follows immediately from unravelling the definitions. □

1.4 Planar crossed simplicial groups

In this section, we introduce a certain class of crossed simplicial groups called *planar crossed simplicial groups*. In terms of the type classification of Table 2, planar crossed simplicial groups can be of cyclic or dihedral type. They correspond, very precisely, to Lie groups that appear as structure groups of surfaces. We start by listing them all in detail.

Example 1.4.1. The **cyclic category** Λ has objects $\langle n \rangle$, $n \geq 0$, while the set of morphisms from $\langle m \rangle$ to $\langle n \rangle$ can be described as follows. Let C_n denote the topological space given by the unit circle in \mathbb{C} equipped with the subset of marked points $\{0, 1, \dots, n\}$, embedded via the map $k \mapsto \exp(2\pi i k / (n+1))$. A morphism $f : \langle n \rangle \rightarrow \langle m \rangle$ in Λ is a homotopy class of monotone maps $C_m \rightarrow C_n$ of degree 1 such that $f(\{0, 1, \dots, m\}) \subset \{0, 1, \dots, n\}$. The category Δ is contained in Λ given by restricting to those morphisms $f : \langle n \rangle \rightarrow \langle m \rangle$ such that the induced map of sets $\{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$ preserves the standard linear orders. The family of groups \mathfrak{G} associated to Λ via (1.1.2) is the family of cyclic groups $\{\mathfrak{G}_n = \mathbb{Z}/(n+1)\}$.

Example 1.4.2. The **dihedral category** Ξ has objects $\overline{\langle n \rangle}$, $n \geq 0$. Morphisms from $\overline{\langle m \rangle}$ to $\overline{\langle n \rangle}$ are homotopy classes of monotone maps $C_m \rightarrow C_n$ of degree ± 1 such that $f(\{0, 1, \dots, m\}) \subset \{0, 1, \dots, n\}$. The family of groups \mathfrak{G} associated to Ξ via (1.1.2) is the family of dihedral groups

$$\mathfrak{G}_n = D_{n+1} = \langle \omega, \tau \mid \omega^2 = 1, \tau^{n+1} = 1, \omega\tau\omega^{-1} = \tau^{-1} \rangle$$

with $|D_{n+1}| = 2(n+1)$.

Example 1.4.3. The **paracyclic category** Λ_∞ has objects \tilde{n} , $n \geq 0$. Morphisms from \tilde{m} to \tilde{n} are maps $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which preserve the standard linear order and satisfy, for every $l \in \mathbb{Z}$, the condition $f(l+m+1) = f(l) + n+1$. The category Δ can be found in Λ_∞ by considering only morphisms $f : \tilde{m} \rightarrow \tilde{n}$ such that

$$f(\{0, 1, \dots, m\}) \subset \{0, 1, \dots, n\}.$$

Construction (1.1.2) yields the constant family of infinite cyclic groups $\{\mathfrak{G}_n = \mathbb{Z}\}$. While this category was introduced in [25], we borrow the terminology *paracyclic* from [29].

Example 1.4.4. The **paradihedral category** Ξ_∞ has objects \hat{n} , $n \geq 0$. Morphisms from \hat{m} to \hat{n} are maps $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which either preserve or reverse the standard linear order and satisfy, for every $l \in \mathbb{Z}$, the condition

$$f(l+m+1) = \begin{cases} f(l) + n+1, & \text{if } f \text{ preserves the order,} \\ f(l) - n-1, & \text{if } f \text{ reverses the order.} \end{cases}$$

Thus $\Xi_\infty \supset \Lambda_\infty \supset \Delta$. Construction (1.1.2) yields the constant family of infinite dihedral groups

$$\mathfrak{G}_n = D_\infty = \langle \omega, \tau \mid \omega^2 = 1, \omega\tau\omega^{-1} = \tau^{-1} \rangle.$$

Example 1.4.5 (Edgewise subdivision). For $N \geq 1$ we have the functor

$$\mathrm{sd}_N : \Delta \longrightarrow \Delta, \quad \mathrm{sd}_N([n-1]) = [Nn-1].$$

A monotone map $\phi : [m] \rightarrow [n]$ is taken to the N -fold concatenation of ϕ with itself. Given a simplicial set X , the N -fold (edgewise) subdivision of X is the new simplicial set $\mathrm{sd}_N^*(X)$ obtained by precomposing $X : \Delta^{\mathrm{op}} \rightarrow \mathcal{S}et$ with sd_N . Geometrically, is obtained by subdividing each simplex of X into several simplices in a regular fashion in such a way that:

- (1) Each edge is subdivided into N intervals by introducing $N-1$ intermediate vertices.
- (2) No other new vertices inside simplices of dimension ≥ 2 are introduced.

In particular, the geometric realizations of X and $\mathrm{sd}_N^* X$ are canonically identified. See [8, §1] [46, E.6.4.3] for more details.

Example 1.4.6. The N -subdivided categories Λ_N, Ξ_N . Given a crossed simplicial group $\Delta\mathfrak{G}$ with the associated simplicial set \mathfrak{G} , one can form the simplicial set $\mathrm{sd}_N^* \mathfrak{G}$ and ask whether it can be completed to a new crossed simplicial group $\Delta \mathrm{sd}_N^* \mathfrak{G}$. It was observed in [25, Ex.7] that this is indeed so for $\Delta\mathfrak{G}$ equal to Λ and Ξ . The corresponded crossed simplicial groups will be denoted Λ_N, Ξ_N . They can be defined explicitly as follows.

(a) The N -cyclic category Λ_N has objects \tilde{n}_N , $n \geq 0$. A morphism from \tilde{m}_N to \tilde{n}_N is an equivalence class of maps $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which preserve the standard linear order and satisfy, for every $l \in \mathbb{Z}$, the condition $f(l+m+1) = f(l) + n+1$. A pair of maps f, g are considered equivalent if there exists an integer r such that $f-g = rN(n+1)$. The category Δ can be found in Λ_N by considering only those morphisms $f : \tilde{m}_N \rightarrow \tilde{n}_N$ such that

$$f(\{0, 1, \dots, m\}) \subset \{0, 1, \dots, n\}.$$

Thus $\mathfrak{G}_n = \{\mathbb{Z}/N(n+1)\mathbb{Z}\}$.

The case $N = 2$ admits a transparent visual description. Let us embed the marked circle C_n from Example 1.4.1 as the middle circle into a *Möbius band* M_n . We view it as a fibration $p_n : M_n \rightarrow C_n$ with fibers being closed intervals. Thus ∂M_n is a circle covering C_n twice. Then elements of $\mathrm{Hom}_{\Lambda_2}(\tilde{m}_2, \tilde{n}_2)$ can be identified with homotopy classes of continuous maps $M_m \rightarrow M_n$ which commute with the projections, send marked points to marked points, boundary to boundary and induce monotone maps $C_m \rightarrow C_n$ of degree ± 1 on the middle circles. For instance, the “elementary rotation” $t_n : M_n \rightarrow M_n$ of M_n along itself at the angle $2\pi/(n+1)$ has order $2(n+1)$, since t_n^{n+1} is the reflection of M_n along C_n .

(b) The N -dihedral category Ξ_N is obtained by modifying the definition of Λ_N , allowing f to either preserve or reverse the standard linear order and imposing the condition on $f(l+m+1)$ as in Example 1.4.4. As in the N -cyclic case, for $N = 2$ we have a topological description in terms of maps of Möbius strips which lift monotone maps of marked circles of degree ± 1 .

Example 1.4.7. The **quaternionic category** ∇ has objects \check{n} , $n \geq 0$. To describe morphisms, we first introduce the *quaternionic groups*

$$Q_n = \langle w, \tau \mid w^2 = \tau^n, \tau^{2n} = w^4 = 1, w\tau w^{-1} = \tau^{-1} \rangle, \quad |Q_n| = 4n.$$

Here $n \geq 1$ is an integer. Thus $Q_1 = \mathbb{Z}/4$. Let \mathbb{H} be the field of quaternions. For any r , we denote by $\mu_r \subset \mathbb{C}^* \subset \mathbb{H}^*$ the group of r th roots of 1. Then Q_n , $n \geq 2$, can be identified with the subgroup in \mathbb{H}^* generated by μ_{2n} and j . In other words, Q_n is, for $n \geq 2$, a finite subgroup in SU_2 of type D in the standard ADE classification (and in that context is also sometimes referred to, confusingly, as the “dihedral group”). Another traditional name for Q_n is the “dicyclic group” [17, §7.2]. We have the central extension

$$(1.4.8) \quad 1 \rightarrow \mathbb{Z}/2 \longrightarrow Q_n \xrightarrow{\pi_n} D_n \rightarrow 1, \quad w \mapsto \omega$$

For $n \geq 3$, D_n embeds into the group of automorphisms of $\mu_n \simeq \mathbb{Z}/n$, while $Q_n \subset \mathbb{H}^*$ normalizes μ_n , so π_n is obtained by looking at the conjugation action of Q_n .

Now, the morphisms of ∇ are defined to be generated by the morphisms in Δ and by elements of the groups $\text{Aut}_{\nabla}(\check{n}) := Q_{n+1}$ subject to the relations spelled out in [46], Prop. 6.3.4(e) (proof). Thus, for ∇ we have

$$\mathfrak{G}_n = Q_{n+1}, \quad |\mathfrak{G}_n| = 4(n+1).$$

Example 1.4.9. The **N -quaternionic category** ∇_N is the crossed simplicial group obtained from ∇ by N -fold subdivision, similarly to Λ_N and Ξ_N . In other words, the simplicial set associated to ∇_N is $(Q_{N(n+1)})_{n \geq 0} = \text{sd}_N^*((Q_{n+1})_{n \geq 0})$, see [25], Ex. 7.

Definition 1.4.10. The crossed simplicial groups from Examples 1.4.1-1.4.9 will be called the *planar crossed simplicial groups*.

Let $\Delta\mathfrak{G}$ be a crossed simplicial group. A functor $X : (\Delta\mathfrak{G})^{\text{op}} \rightarrow \mathcal{C}$ with values in a category \mathcal{C} will be called a $\Delta\mathfrak{G}$ -object of \mathcal{C} . Via the embedding $i : \Delta \rightarrow \Delta\mathfrak{G}$, we can describe X as the simplicial object i^*X equipped with additional structure: For every $n \geq 0$, the object X_n carries an action of the group $(\mathfrak{G}_n)^{\text{op}}$, compatible with the simplicial structure according to the relations in $\Delta\mathfrak{G}$. In the case when $\Delta\mathfrak{G} = \Lambda, \Xi, \Lambda_\infty$ etc. is planar, we will speak about *cyclic*, *dihedral*, *paracyclic* etc. objects in \mathcal{C} .

For example, the simplicial set \mathfrak{G} extends naturally to a $\Delta\mathfrak{G}$ -object, as it follows from Example 1.1.6.

Example 1.4.11. A paracyclic object $X : (\Lambda_\infty)^{\text{op}} \rightarrow \mathcal{C}$ can be described as a simplicial object i^*X equipped with automorphisms $t_n : X_n \rightarrow X_n$, $n \geq 0$, satisfying the relations

$$\partial_i t_n = \begin{cases} \partial_n & \text{for } i = 0, \\ t_{n-1} \partial_{i-1} & \text{for } 1 \leq i \leq n, \end{cases} \quad s_i t_n = \begin{cases} t_{n+1}^2 s_n & \text{for } i = 0, \\ t_{n+1} s_{i-1} & \text{for } 1 \leq i \leq n. \end{cases}$$

An N -cyclic object admits an identical description with the additional condition that the automorphism t_n , $n \geq 0$, has order $N(n+1)$. In particular, for $N = 1$ we get a characterization of cyclic objects.

1.5 Relation to planar Lie groups

1.5.A. Connective coverings and 2-groups

In this paper, all topological groups will be assumed to have the homotopy type of a CW-complex. Let G be a topological group. We denote by $G_e \subset G$ the connected component of the identity. The two structures on G give rise to the following algebraic data:

- (1) The possibly nonabelian group $P = \pi_0(G)$.
- (2) The abelian group $A = \pi_1(G_e, e)$ equipped with a natural P -action induced by conjugation in G .
- (3) The cohomology class $\gamma = \gamma_G \in H^3(P, A)$ defined as follows. For every connected component $a \in P = \pi_0(G)$, we choose a representative $g_a \in G$. For every pair $a, b \in P$, we choose a path $\xi_{a,b}$ joining $g_a g_b$ and g_{ab} . We denote by $\xi_{a,b}^{-1}$ the same path run in the opposite direction. Then, for every triple $a, b, c \in P$, we have a loop in the component abc :

$$\begin{array}{ccc}
 g_a g_b g_c & \xrightarrow{\xi_{a,b} \cdot g_c} & g_{ab} g_c \\
 \uparrow g_a \cdot \xi_{b,c}^{-1} & & \downarrow \xi_{ab,c} \\
 g_a g_{bc} & \xleftarrow{g_a \cdot \xi_{a,b}^{-1}} & g_{abc}
 \end{array}$$

By multiplying this loop pointwise on the left with g_{abc}^{-1} , we obtain a loop $L(a, b, c)$ in G_e based at e . We denote by $\gamma(a, b, c) \in A = \pi_1(G_e, e)$ the class of $L(a, b, c)$. The collection $\{\gamma(a, b, c)\}$ forms a 3-cocycle of P with coefficients in A whose cohomology class is independent on the choices.

Definition 1.5.1. A morphism $p : \tilde{G} \rightarrow G$ of topological groups is called a *connective covering*, if p is a covering of its image, and $p^{-1}(G_e)$ is connected, that is, coincides with \tilde{G}_e . A *proper covering* is a surjective connective covering.

The condition $p^{-1}(G_e) = \tilde{G}_e$ is equivalent to saying that the induced homomorphism $p_* : \pi_0(\tilde{G}) \rightarrow \pi_0(G)$ is injective.

We denote by $\text{Con}(G)$ the category of proper coverings of G , with morphisms being morphisms of topological groups commuting with the projections to G . Any $\tilde{G} \in \text{Con}(G)$ gives rise to subgroups

$$\begin{aligned}
 (1.5.2) \quad \tilde{P} &= p_*(\pi_0(\tilde{G})) \subset P = \pi_0(G), \\
 \tilde{A} &= \pi_1(\tilde{G}_e, e) \subset A = \pi_1(G_e, e),
 \end{aligned}$$

with \tilde{A} being P -invariant.

The structure of connective coverings of a given G is classically known to depend on the algebraic data (P, A, γ) above. We refer to [63, 12] for detailed background and formulate here the following summary result.

Proposition 1.5.3. (a) Suppose we are given subgroups $\tilde{P} \subset P$, $\tilde{A} \subset A$ so that \tilde{A} is P -invariant. For the existence of a connective covering \tilde{G} realizing \tilde{P} and \tilde{A} as in (1.5.2), it is necessary and sufficient that the image of γ_G in $H^3(\tilde{P}, A/\tilde{A})$ be 0.

(b) In particular, G has a “universal covering” (a proper covering \tilde{G} with \tilde{G}_e simply connected), if and only if $\gamma_G = 0$.

(c) If $f : G_1 \rightarrow G_2$ is a morphism of topological groups which is a homotopy equivalence of topological spaces, then pullback under f gives an equivalence of categories

$$f^* : \text{Con}(G_2) \longrightarrow \text{Con}(G_1). \quad \square$$

We note that algebraic data (P, A, γ) as above, appear in the classification of the following related types of objects:

- (1) **Connected homotopy 2-types**, i.e., homotopy types of connected CW-complexes X with $\pi_{>2}(X, x) = 0$. In this context, $P = \pi_1(X, x)$, $A = \pi_2(X, x)$. A topological group G produces such as X by forming the classifying space BG and then killing $\pi_{>2}$.
- (2) **2-groups** (also known as *non-symmetric Picard categories*). By definition, a 2-group is an essentially small monoidal category $(\mathbf{G}, \otimes, \mathbf{1})$ in which each object is invertible (up to isomorphism) with respect to \otimes and each morphism is invertible with respect to composition. In this case

$$P = \text{Pic}(\mathbf{G}) = (\text{Ob}(\mathbf{G})/\text{iso}, \otimes), \quad A = \text{Aut}_{\mathbf{G}}(\mathbf{1}).$$

A topological group G produces a 2-group $\mathbf{G} = \Pi_1(G)$, the fundamental groupoid with all $g \in G$ taken as base points, with \otimes given by the product in G . Conversely, a 2-group \mathbf{G} has a realization (loop space of the classifying space) $|\mathbf{G}|$ which can be realized as a topological group with $\pi_0 = G$, $\pi_1 = A$.

- (3) **Crossed modules**, i.e., data consisting of a morphism of groups

$$\{K^{-1} \xrightarrow{\partial} K^0\}$$

and an action $\epsilon : K^0 \rightarrow \text{Aut}(K^{-1})$ satisfying the axioms

$$\begin{aligned} \epsilon(\partial(k_{-1}))(k'_{-1}) &= k'_{-1} \cdot k_{-1} \cdot (k'_1)^{-1}, \quad k_{-1}, k'_{-1} \in K^{-1}, \\ \partial(k_{-1}) \cdot k_0 \cdot \partial(k_{-1})^{-1} &= \partial(\epsilon(k_0)(k_{-1})), \quad k_0 \in K^0, k_{-1} \in K^{-1}. \end{aligned}$$

These axioms ensure that $\text{Im}(\partial)$ is a normal subgroup in K^0 , so we have the group $P = K^0/\text{Im}(\partial)$, and that $A = \text{Ker}(\partial)$ lies in the center of K^{-1} , in particular, A is abelian (and P acts on A via ϵ).

A topological group G gives a crossed module by taking $K^0 = G$ and $K^{-1} = (G_e)^\sim$ to be the universal covering of G_e with base point e , see [12] and references therein. Note the similarity between the above axioms of a crossed module and the identities holding in any crossed simplicial group (Proposition 1.1.4(c)).

1.5.B. Planar Lie groups

Let S^1 be the standard unit circle in \mathbb{C} , and $\text{Homeo}(S^1)$ be the group of all self-homeomorphisms of S^1 with the standard (compact-open) topology. We have morphisms of topological groups

$$O(2) \hookrightarrow GL(2, \mathbb{R}) \longrightarrow \text{Homeo}(S^1)$$

which are homotopy equivalences, Therefore, by Proposition 1.5.3(c), the categories of connective coverings of these three groups are naturally identified.

Connective coverings of $O(2)$ will be called (*thin*) *planar Lie groups*. We will consider them as basic objects notation-wise and will denote them by standard Roman letters such as G .

Connective coverings of $GL(2, \mathbb{R})$ will be called *thick planar Lie groups*. We will denote them by G° , where G is a thin planar Lie group. For example, $SO(2)^\circ = GL^+(2, \mathbb{R})$ is the subgroup of matrices with positive determinant.

Connective coverings of $\text{Homeo}(S^1)$ will be called *planar homeomorphism groups*. We will denote them by $\text{Homeo}^G(S^1)$, where G is as above.

We recall the well known classification of planar Lie groups which is an analog of the theory of Pin groups [5] extending $O(d)$ by $\mathbb{Z}/2$ for any d . Remarkably, the groups in this classification correspond very precisely to planar crossed simplicial groups. We start with describing the examples and formulating the correspondence.

Examples 1.5.4. (a) The group $SO(2)$ corresponds to the cyclic category Λ .

(b) The group $O(2)$ corresponds to the dihedral category Ξ .

(c) We denote by $\widetilde{SO}(2) \rightarrow SO(2)$ the universal covering of $SO(2) \simeq S^1$. It corresponds to the paracyclic category Λ_∞ .

(d) We denote

$$\widetilde{O}(2) = (\mathbb{Z}/2) \ltimes \mathbb{R} \longrightarrow O(2) = (\mathbb{Z}/2) \ltimes S^1$$

the proper covering of the semi-direct product induced from the universal covering $\mathbb{R} \rightarrow S^1$. It corresponds to the paradihedral category Ξ_∞ .

(e) Consider the group

$$P = \{q \in \mathbb{H}^* \mid |q| = 1, qS^1q^{-1} \subset S^1\}.$$

Here S^1 is considered as the unit circle subgroup in $\mathbb{C}^* \subset \mathbb{H}^*$. In other words, P is the normalizer of a maximal torus in $SU(2)$. Like $O(2)$, the group P has $\pi_0 = \mathbb{Z}/2$ and $P_e = S^1$, but the corresponding extension of $\mathbb{Z}/2$ by S^1 is different from the one provided by $O(2)$. Similarly to (1.4.8), looking at the conjugation action gives a central extension

$$1 \rightarrow \mathbb{Z}/2 \longrightarrow P \xrightarrow{\pi} O(2) \rightarrow 1$$

with π a proper covering. This makes P into a planar Lie group. It corresponds to the quaternionic category ∇ .

(f) Let $N \geq 1$. We denote the unique N -fold proper covering of $SO(2)$ by

$$1 \rightarrow \mathbb{Z}/N \longrightarrow \text{Spin}_N(2) \xrightarrow{\phi_N} SO(2) \rightarrow 1.$$

As an abstract Lie group, $\text{Spin}_N(2)$ is identified with $SO(2)$, and ϕ_N is the N th power homomorphism. The group $\text{Spin}_N(2)$ corresponds to the N -cyclic category Λ_N .

(g) Let us represent elements of $O(2) = (\mathbb{Z}/2) \ltimes S^1$ as pairs $(\alpha, z) = \alpha \cdot z$ with $\alpha \in \mathbb{Z}/2$, $z \in S^1$, so that $\alpha z \alpha^{-1} = z^{-1}$. Then the map

$$f_N : O(2) \longrightarrow O(2), \quad (\alpha, z) \mapsto (\alpha, z^N).$$

is a (continuous) homomorphism and, moreover, a proper covering with kernel \mathbb{Z}/N . We denote this covering

$$1 \rightarrow \mathbb{Z}/N \longrightarrow \text{Pin}_N^+(2) \xrightarrow{p_N^+} O(2) \rightarrow 1$$

Thus $\text{Pin}_N^+(2) = O(2)$ as an abstract Lie group, and $p_N^+ = f_N$. The planar Lie group $\text{Pin}_N^+(2)$ corresponds to the N -dihedral category Ξ_N .

(h) If $N = 2M$ is even, then the composition $P \xrightarrow{\pi} O(2) \xrightarrow{f_M} O(2)$ is a proper covering with kernel \mathbb{Z}/N . We denote this covering

$$1 \rightarrow \mathbb{Z}/N \longrightarrow \text{Pin}_N^-(2) \xrightarrow{p_N^-} O(2) \rightarrow 1$$

Thus $\text{Pin}_N^-(2) = Q_\infty$ as an abstract Lie group, and $p_N^- = f_M \pi$. The planar Lie group $\text{Pin}_N^-(2)$ corresponds to the M -quaternionic category ∇_M .

Proposition 1.5.5. *The groups from Examples 1.5.4 exhaust all planar Lie groups up to isomorphism.* \square

Proof: The question reduces to classification of proper coverings of $O(2)$ for which see, e.g., [63, §8]. \square

The correspondence between crossed simplicial groups and planar Lie groups are collected in the first three columns of Table 3. The fourth column means that a crossed simplicial group of Lie type “governs” the combinatorial topology of the corresponding class of structured surfaces. This will be explained in more details in §4.1. The last column will be explained in §1.6 and 1.7.

We further arrange planar Lie groups and natural homomorphisms between them on the left in Table 4 and observe that these homomorphisms are matched by natural functors between corresponding crossed simplicial groups. Each homomorphism between Lie groups in the table is either a surjection, in which case we label the arrow with the kernel of the surjection, or is an inclusion of a normal subgroup with quotient $\mathbb{Z}/2$. Note that the group $\text{Pin}_{2M}^-(2)$, isomorphic to $P \subset \mathbb{H}^*$, does not admit a simply-connected proper covering [63, §8].

CS group $\Delta\mathfrak{G}$	Sequence of groups \mathfrak{G}_n	Lie group $G = \mathfrak{G} $	Type of G^\diamond -structured surfaces	Extra data on associative algebras and categories
Cyclic category Λ	$\mathbb{Z}/(n+1)$	$SO(2)$	Oriented surfaces	No data
Dihedral category Ξ	D_{n+1}	$O(2)$	Unoriented surfaces	Anti-automorphism of order 2
Paracyclic category Λ_∞	$\mathfrak{G}_n = \mathbb{Z}$	$\widetilde{SO}(2)$	Framed surfaces	Automorphism
Paradihedral category Ξ_∞	$\mathfrak{G}_n = D_\infty$	$\widetilde{O}(2)$	Surfaces with framing on orientation cover	Automorphism t and involution ω s.t. $\omega t \omega = t^{-1}$.
N -cyclic category Λ_N	$\mathbb{Z}/N(n+1)$	$\text{Spin}(2)_N$	N -spin (oriented) surfaces	Automorphism of order N
N -dihedral category Ξ_N	$D_{N(n+1)}$	$\text{Pin}_N^+(2)$	Unoriented, with $\text{Pin}_N^+(2)$ -structure	Twisted action of D_N .
M -quaternionic category ∇_M	$Q_{M(n+1)}$	$\text{Pin}_{2M}^-(2)$	Unoriented, with $\text{Pin}_{2M}^-(2)$ -structure	Twisted action of Q_M

Table 3: Planar crossed simplicial group and associated structures.

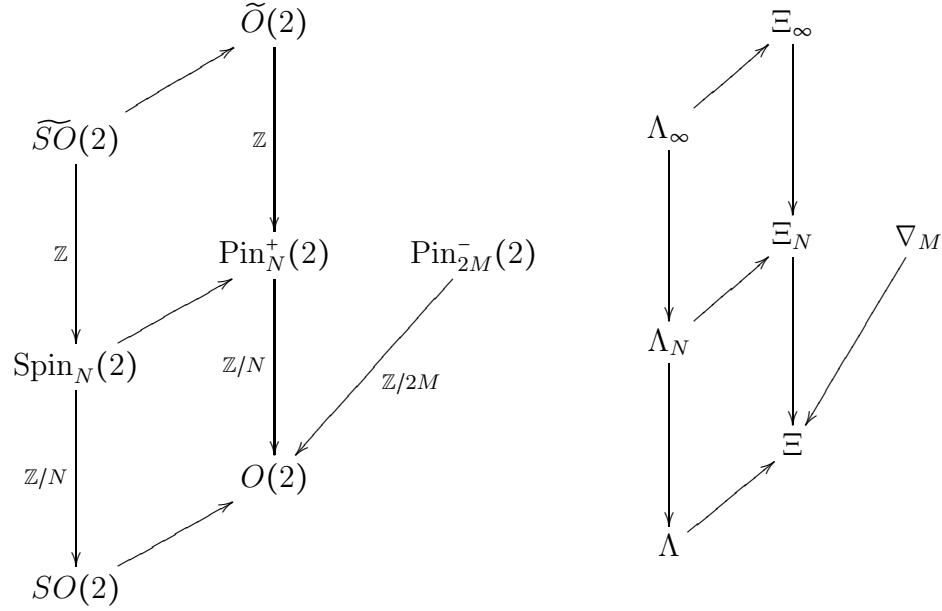


Table 4: Correspondence between planar crossed simplicial groups and planar Lie groups.

The following statement materializes the correspondence and systematizes the results of several authors.

Theorem 1.5.6. (a) *Let $\Delta\mathfrak{G}$ be any crossed simplicial group. Then the geometric realization $|\mathfrak{G}|$ of the simplicial set \mathfrak{G} has a natural group structure, making it a topological group. Further, the geometric realization $|N(\Delta\mathfrak{G})|$ of the nerve of $\Delta\mathfrak{G}$ is homotopy equivalent to the classifying space $B|\mathfrak{G}|$.*

(b) *Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group. Then:*

(b1) *$|\mathfrak{G}| = G$ is the planar Lie group corresponding to $\Delta\mathfrak{G}$ in Table 3. In particular, there is a covering $p: G \rightarrow O(2)$.*

(b2) *The group \mathfrak{G}_n is identified with the subgroup $p^{-1}(D_{n+1})$, where D_{n+1} is embedded into $O(2) = (\mathbb{Z}/2) \ltimes S^1$ as $(\mathbb{Z}/2) \ltimes \mu_{n+1}$.*

(c) *Let $X: \Delta\mathfrak{G}^{\text{op}} \rightarrow \text{Set}$ be any $\Delta\mathfrak{G}$ -set. Then $|i^*X|$, the geometric realization of X considered as a simplicial set, has a natural G -action.*

(d) *The homotopy category of G -spaces can be identified with the localization of the category $\text{Fun}(\Delta\mathfrak{G}^{\text{op}}, \text{Set})$ along morphisms which give homotopy equivalences of the induced simplicial sets.*

Proof. (a) was shown by Fiedorowicz and Loday [25]. The identification of $|\mathfrak{G}|$ in the examples of (b) is also immediate from their considerations. The remaining parts are well known [21] [46, Ch.7] in the cyclic and paracyclic cases $\Delta\mathfrak{G} = \Lambda, \Lambda_\infty$, and were extended

by Dunn [20] to the dihedral and quaternionic cases. The N -subdivided cases can be deduced from these using the fact that sd_N^* does not change the geometric realization. The paradihedral case can be deduced from the paracyclic one by identifying Ξ_∞ with the semi-direct product of $\mathbb{Z}/2$ and Λ_∞ . We leave the remaining details to the reader. \square

1.6 Structured nerves

In this and the following section, we explain the last column of Table 3 which puts planar crossed simplicial groups in correspondence with algebras and categories with additional types of extra data such as involutions, automorphisms and so on. For this, we systematize and expand the ideas of Loday [46, Ch. 5] about “variations on cyclic homology”.

1.6.A. Canonical parity and the $\Delta\mathfrak{G}$ -nerve

Recall (Example 1.3.3) that any crossed simplicial group $\Delta\mathfrak{G}$ contains the semiconstant crossed simplicial group $\Delta\{\mathfrak{G}_0\}$ generated by \mathfrak{G}_0 . By Proposition 1.3.6, the group \mathfrak{G}_0 acts on Δ and is hence equipped with a *canonical parity* $\mathfrak{G}_0 \rightarrow \text{Aut}(\Delta) \cong \mathbb{Z}/2\mathbb{Z}$. We explicitly describe the canonical parity of \mathfrak{G}_0 for planar crossed simplicial groups.

Proposition 1.6.1. *Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group with the corresponding planar Lie group $p : G \rightarrow O(2)$. Then:*

- (a) *Let $O(1) = \mathbb{Z}/2$ be embedded into $O(2)$ in the standard way. Then \mathfrak{G}_0 is identified, as a group with parity, with the preimage of $O(1)$ in G , so we have the Cartesian square*

$$\begin{array}{ccc} \mathfrak{G}_0 & \xrightarrow{\epsilon} & G \\ \rho \downarrow & & \downarrow p \\ \mathbb{Z}/2 = O(1) & \longrightarrow & O(2). \end{array}$$

- (b) *Explicitly, the canonical parities of $\mathfrak{G}_0 = \mathbb{Z}, (\mathbb{Z}/2) \ltimes \mathbb{Z}, \mathbb{Z}/N, D_N, Q_N$ are given as follows: the groups \mathbb{Z} and \mathbb{Z}/N are equipped with the trivial homomorphism to $\mathbb{Z}/2$, and the group D_N and Q_N with homomorphisms*

$$\begin{aligned} \rho_D : D_N &\longrightarrow \mathbb{Z}/2, & \rho_D(\omega) &= \bar{1}, & \rho_D(\tau) &= \bar{0}, \\ \rho_Q : Q_N &\longrightarrow \mathbb{Z}/2, & \rho_Q(w) &= \bar{1}, & \rho_Q(\tau) &= \bar{0}, \end{aligned}$$

and $(\mathbb{Z}/2) \ltimes \mathbb{Z}$ with the projection to $\mathbb{Z}/2$.

Proof. We recall that G is identified with $|\mathfrak{G}|$ by Theorem 1.5.6(b1). A homomorphism $\epsilon : \mathfrak{G}_0 \rightarrow |\mathfrak{G}|$ is obtained from the compatible system of homomorphisms $\omega_n^* : \mathfrak{G}_0 \rightarrow \mathfrak{G}_n$. The verification that ϵ is an embedding with the required properties, is obtained using case by case analysis. \square

Let $\Delta\mathfrak{G}$ be any crossed simplicial group, and let $n \geq 0$. We call the representable $\Delta\mathfrak{G}$ -set

$$\Delta\mathfrak{G}^n := \text{Hom}_{\Delta\mathfrak{G}}(-, [n]) : \Delta\mathfrak{G}^{\text{op}} \longrightarrow \text{Set}$$

the n -dimensional $\Delta\mathfrak{G}$ -simplex. In particular, for Λ , Ξ , etc, we will speak about *cyclic simplices*, *dihedral simplices*, etc. As an immediate corollary of the results of Section 1.3, we have the following statement.

Proposition 1.6.2. *For any crossed simplicial group $\Delta\mathfrak{G}$, we have a natural functor*

$$\Delta\mathfrak{G} \longrightarrow \mathfrak{G}_0 - \text{Cat}, [n] \mapsto \text{FC}(\Delta\mathfrak{G}^n|_{\Delta})$$

where

$$\mathfrak{G}_0 - \text{Cat} = \text{Fun}_{\mathcal{B}\mathbb{Z}/2\mathbb{Z}}(\mathcal{B}\mathfrak{G}_0, \mathbb{Z}/2\mathbb{Z} \ltimes \text{Cat})$$

denotes the category of small categories equipped with a twisted action of \mathfrak{G}_0 .

Proof. Denoting Res the restriction from $\Delta\mathfrak{G}$ to $\Delta\{\mathfrak{G}_0\}$, we obtain a sequence of functors

$$\Delta\mathfrak{G} \xrightarrow{\text{Yoneda}} \text{Set}_{\Delta\mathfrak{G}} \xrightarrow{\text{Res}} \text{Set}_{\Delta\{\mathfrak{G}_0\}} \xrightarrow{1.3.13} \mathfrak{G}_0 - \text{Set}_{\Delta} \xrightarrow{\text{FC}} \mathfrak{G}_0 - \text{Cat}$$

whose composite is the desired functor. \square

As a consequence, we obtain the following result.

Theorem 1.6.3. *Let $\Delta\mathfrak{G}$ be any crossed simplicial group. Equip \mathfrak{G}_0 with its canonical parity. Then to each small category \mathcal{C} with a twisted \mathfrak{G}_0 -action, there is an associated $\Delta\mathfrak{G}$ -set $N^{\Delta\mathfrak{G}}(\mathcal{C})$ called the $\Delta\mathfrak{G}$ -nerve of \mathcal{C} such that*

$$N^{\Delta\mathfrak{G}}(\mathcal{C})_n = \text{Hom}_{\mathfrak{G}_0 - \text{Cat}}(\text{FC}(\Delta\mathfrak{G}^n|_{\Delta}), \mathcal{C}).$$

Proof. This follows by postcomposing the functor of Proposition 1.6.2 with $\text{Hom}_{\mathfrak{G}_0 - \text{Cat}}(-, \mathcal{C})$. \square

1.6.B. Examples

We now give a more explicit description of $N^{\Delta\mathfrak{G}}(\mathcal{C})$ for each planar $\Delta\mathfrak{G}$.

Cyclic case (Connes). Here $\mathfrak{G}_0 = \mathbb{Z}/1$ is the trivial group. Let \mathcal{C} be a small category. Its *cyclic nerve* is the simplicial set $\text{NC}_{\bullet}\mathcal{C}$ with $\text{NC}_n\mathcal{C}$ being the set of diagrams

$$(1.6.4) \quad x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \dots \xrightarrow{a_{n-1}} x_n \xrightarrow{a_n} x_0.$$

As well known, the cyclic rotation of such diagrams makes $\text{NC}_{\bullet}\mathcal{C}$ into a cyclic set.

Dihedral case (Loday). Here $\mathfrak{G}_0 = D_1 \simeq \mathbb{Z}/2$ with the identity homomorphism to $\mathbb{Z}/2$, so a twisted \mathfrak{G}_0 -action is an involution.

Let $(\mathcal{C}, *)$ be a small category with involution, which we write as an isomorphism $*$: $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$, $** = \text{Id}$. Then $\text{NC}_{\bullet} \mathcal{C}$ becomes a dihedral set, with the generator $\omega_n \in D_{n+1} = \text{Aut} \langle n \rangle$ sending a diagram (1.6.4) into

$$x_1^* \xrightarrow{a_0^*} x_0^* \xrightarrow{a_n^*} x_n^* \xrightarrow{a_{n-1}^*} x_{n-1}^* \xrightarrow{a_{n-2}^*} \dots \xrightarrow{a_2^*} x_2^* \xrightarrow{a_1^*} x_1^*.$$

Paracyclic and N -cyclic cases. In the paracyclic case $\mathfrak{G}_0 = \mathbb{Z}$ with trivial homomorphism into $\mathbb{Z}/2$, so a twisted \mathfrak{G}_0 -action is an automorphism. In the N -cyclic case a twisted $\mathfrak{G}_0 = \mathbb{Z}/N$ -action is an automorphism of order N .

Let \mathcal{C} be a category equipped with a (covariant) automorphism F . We have then the *twisted cyclic nerve* $\text{NC}_{\bullet}^F \mathcal{C}$, see [22]. This is a simplicial set whose n -simplices are diagrams

$$(1.6.5) \quad x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \dots \xrightarrow{a_{n-1}} x_n \xrightarrow{a_n} F(x_0).$$

We have the transformation $\tau_n : \text{NC}_n^F \mathcal{C} \rightarrow \text{NC}_n^F \mathcal{C}$ sending such a diagram to

$$(1.6.6) \quad F^{-1}(x_n) \xrightarrow{F^{-1}(a_n)} x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \dots \xrightarrow{a_{n-2}} x_{n-1} \xrightarrow{a_{n-1}} x_n = F(F^{-1}(x_n)).$$

One sees directly that the τ_n make $\text{NC}_{\bullet}^F \mathcal{C}$ into a paracyclic set. If $F^N = \text{Id}$ for some N , then $\tau_n^{N(n+1)} = \text{Id}$ as well, and so $\text{NC}_{\bullet}^F \mathcal{C}$ is an N -cyclic set.

The twisted cyclic nerve is the categorical analog of the *twisted loop space* $L^F X$ associated to a topological space X together with a homeomorphism $F : X \rightarrow X$. Explicitly, $L^F X$ consists of continuous maps $\gamma : \mathbb{R} \rightarrow X$ such that $\gamma(t+1) = F(\gamma(t))$.

Quaternionic case (Dunn): $\mathfrak{G}_0 = \mathbb{Z}/4$. Quaternionic homology was originally introduced by Loday [45] [46, §5.2] as a purely technical modification of dihedral homology in the case when 2 is not invertible in the base ring. However, from our perspective, the more natural framework for quaternionic homology is provided by algebras and categories with a *twisted action* of $\mathfrak{G}_0 = Q_1 = \mathbb{Z}/4$, i.e., with an *anti-automorphism* J of order 4. In this case J^2 is an automorphism of order 2. This point of view goes back to Dunn [20, §2].

More precisely, let \mathcal{C} be a category and $J : \mathcal{C} \rightarrow \mathcal{C}$ be a contravariant functor such that $J^4 = \text{Id}$. Consider the 2-cyclic set $\text{NC}_{\bullet}^{J^2} \mathcal{C}$. Define $w = w_n : \text{NC}_n^{J^2} \mathcal{C} \rightarrow \text{NC}_n^{J^2} \mathcal{C}$ by

$$w_n \{ x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \dots \xrightarrow{a_{n-1}} x_n \xrightarrow{a_n} J^2 x_0 \} = \\ \{ J^{-1} x_1 \xrightarrow{J^{-1}(a_0)} J^{-1}(x_0) \xrightarrow{J(a_n)} J(x_n) \xrightarrow{J(a_{n-1})} J(x_{n-1}) \xrightarrow{J(a_{n-2})} \dots \xrightarrow{J(a_1)} J(x_1) = J^2(J^{-1} x_1) \}.$$

(Note that the source of $J(a_n)$ is $J(J^2 x_0) = J^{-1} x_0$.) Then w_n together with τ_n defined as in (1.6.6), define an action of Q_{n+1} on $\text{NC}_n^{J^2} \mathcal{C}$. One verifies that in this way $\text{NC}_{\bullet}^{J^2} \mathcal{C}$ becomes a quaternionic set.

N -dihedral case: $\mathfrak{G}_0 = D_N$. Let \mathcal{C} be a category with a twisted action of D_N , i.e., with a contravariant functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ and a covariant functor $T : \mathcal{C} \rightarrow \mathcal{C}$ satisfying the

relations of D_N . In particular, $T^N = \text{Id}$, so we have an N -cyclic set $\text{NC}_\bullet^T \mathcal{C}$. For a given n , the set $\text{NC}_n^T \mathcal{C}$ is acted upon by the transformation τ_n as in (1.6.6) and ω_n which takes an n -simplex

$$x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} x_n \xrightarrow{a_n} T(x_0)$$

into

$$\Omega(x_1) \xrightarrow{\Omega(a_0)} \Omega(x_0) \xrightarrow{T\Omega(a_n)} T\Omega(x_n) \xrightarrow{T\Omega(a_{n-1})} T\Omega(x_{n-1}) \xrightarrow{T\Omega(a_{n-2})} \cdots \xrightarrow{T\Omega(a_1)} T\Omega(x_1).$$

Note that by the dihedral relations we can write the source of $\Omega(a_n)$ as $\Omega T x_0 = T^{-1} \Omega x_0$, so $T\Omega(a_n) : \Omega x_0 \rightarrow T\Omega x_n$. The maps τ_n and ω_n satisfy the relations of $D_{N(n+1)}$ and one checks directly that in this way $\text{NC}_\bullet^T \mathcal{C}$ becomes an N -dihedral set.

N -quaternionic case: $\mathfrak{G}_0 = Q_N$. Let \mathcal{C} be a category with a twisted action of Q_N , i.e., with a contravariant functor $W : \mathcal{C} \rightarrow \mathcal{C}$ and a covariant functor $T : \mathcal{C} \rightarrow \mathcal{C}$, satisfying the relations of Q_N . Then $T^{2N} = \text{Id}$, so we have the $2N$ -cyclic set $\text{NC}_\bullet^T \mathcal{C}$. We define the action of $Q_{N(n+1)}$ on $\text{NC}_n^T \mathcal{C}$ by transformations τ_n , defined as in (1.6.6), and w_n which takes a simplex

$$x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} x_n \xrightarrow{a_n} T(x_0)$$

into

$$W(x_1) \xrightarrow{W(a_0)} W(x_0) \xrightarrow{TW(a_n)} TW(x_n) \xrightarrow{TW(a_{n-1})} TW(x_{n-1}) \longrightarrow \cdots \xrightarrow{TW(a_1)} TW(x_1).$$

As before, the identity $WTW^{-1} = T^{-1}$ implies that $TW(a_n)$ acts from $W(x_0)$ to $TW(x_n)$. The relations of the quaternionic group $Q_{N(n+1)}$ and the compatibility of the actions of the $Q_{N(n+1)}$, $n \geq 0$ and of Δ are verified directly.

1.7 Structured Hochschild complexes and $\Delta\mathfrak{G}$ -Frobenius algebras

Let \mathbf{k} be a field. By a *vector space* (resp. *algebra*) we will always mean a \mathbf{k} -vector space, resp. a \mathbf{k} -algebra. We denote by Vect the category of vector spaces.

1.7.A. The $\Delta\mathfrak{G}$ -structure on the Hochschild complex

We have the following enriched variant of Theorem 1.6.3 which we will only formulate for algebras (instead of more general \mathbf{k} -linear categories).

Theorem 1.7.1. *Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group. Let A be a unital associative algebra with a twisted action of \mathfrak{G}_0 . Then the collection of vector spaces*

$$C_\bullet(A) = \{C_n(A) = A^{\otimes(n+1)}\}_{n \geq 0},$$

given by the terms of the Hochschild complex of A , naturally forms a $\Delta\mathfrak{G}$ -vector space. In particular, the group \mathfrak{G}_n acts on $C_n(A)$.

Proof. The cosimplicial structure is obtained by letting face maps act by insertion of 1 and degeneracy maps act by multiplication in A . Let us construct an action of \mathfrak{G}_n on $A^{\otimes(n+1)}$. Consider the wreath product

$$S_{n+1} \wr \mathfrak{G}_0 := S_{n+1} \ltimes \mathfrak{G}_0^{n+1},$$

where the action of S_{n+1} on \mathfrak{G}_0^{n+1} is by permuting the Cartesian factors. Note that we have a projection map $S_{n+1} \wr \mathfrak{G}_0 \rightarrow S_{n+1}$. We now claim that the homomorphism $\lambda_n : \mathfrak{G}_n \rightarrow S_{n+1}$ from Proposition 1.1.7 has a natural lifting to a homomorphism

$$L_n : \mathfrak{G}_n \longrightarrow S_{n+1} \wr \mathfrak{G}_0.$$

Indeed, by the canonical factorization

$$\mathrm{Hom}_{\Delta\mathfrak{G}}([0], [n]) = \mathfrak{G}_0 \times \mathrm{Hom}_{\Delta}([0], [n]),$$

and \mathfrak{G}_n acts on this set on the left by composition of morphisms in $\Delta\mathfrak{G}$. We see, first, that the action of any $g \in \mathfrak{G}_n$ takes any fiber of the projection to $\mathrm{Hom}_{\Delta}([0], [n])$ to another fiber. The induced permutation of these fibers is precisely $\lambda_n(g) \in S_{n+1}$. Second, if we identify each fiber with \mathfrak{G}_0 , then the identification of one fiber with another is given by the left multiplication with some element of \mathfrak{G}_0 . These elements, together with $\lambda_n(g)$, give the desired $L_n(g) \in S_{n+1} \wr \mathfrak{G}_0$.

Since \mathfrak{G}_0 acts on A as a vector space, we have the natural action of $S_{n+1} \wr \mathfrak{G}_0$ on $A^{\otimes(n+1)}$ which combines the permutations of the tensor factors and action of \mathfrak{G}_0 on each factor. We pull this action back along L_n to get a \mathfrak{G}_n -action on $A^{\otimes(n+1)}$. The compatibility of these actions with the simplicial structure is verified straightforwardly. \square

Remark 1.7.2. The Hochschild complex of an associative algebra is usually constructed as the totalization of a *simplicial* \mathbf{k} -vector space. This simplicial object can be obtained by precomposing the functor $\Delta\mathfrak{G} \rightarrow \mathrm{Vect}$ from Theorem 1.7.1 with

$$\Delta^{\mathrm{op}} \longrightarrow \Delta\mathfrak{G}^{\mathrm{op}} \xrightarrow{D} \Delta\mathfrak{G}$$

where D denotes a self-duality of the category $\Delta\mathfrak{G}$ (cf. Corollary 2.1.20 below).

1.7.B. $\Delta\mathfrak{G}$ -(co)homology, traces and Frobenius algebras

We denote by $\Delta\mathfrak{G} - \mathrm{Vect}$ the category of $\Delta\mathfrak{G}$ -vector spaces. Let $\underline{\mathbf{k}}$ be the constant $\Delta\mathfrak{G}$ -vector space with stalk \mathbf{k} , i.e., the contravariant functor which takes each object to \mathbf{k} and each morphism to the identity.

Let A be a unital associative algebra with a twisted \mathfrak{G}_0 -action. By Theorem 1.7.1, $C_{\bullet}(A)$ is an object of $\Delta\mathfrak{G} - \mathrm{Vect}$. Following the approach of Connes and Loday, we define the $\Delta\mathfrak{G}$ -homology and cohomology of A to be

$$\begin{aligned} H_{\bullet}^{\Delta\mathfrak{G}}(A) &= \mathrm{Tor}_{\bullet}^{\Delta\mathfrak{G} - \mathrm{Vect}}(\underline{\mathbf{k}}, C_{\bullet}(A)), \\ H_{\Delta\mathfrak{G}}^{\bullet}(A) &= \mathrm{Ext}_{\Delta\mathfrak{G} - \mathrm{Vect}}^{\bullet}(C_{\bullet}(A), \underline{\mathbf{k}}) \end{aligned}$$

where \mathbf{k} denotes the constant $\Delta\mathfrak{G}$ -module.

Example 1.7.3. Elements of $H_{\Delta\mathfrak{G}}^0(A) = \text{Hom}_{\Delta\mathfrak{G}\text{-Vect}}(C_\bullet(A), \mathbf{k})$ will be called $\Delta\mathfrak{G}$ -traces on A . Thus a $\Delta\mathfrak{G}$ -trace β is a collection of multilinear forms $\beta_n : A^{\otimes(n+1)}, \mathbf{k}$, $n \geq 0$. The condition that β is a morphism of cosimplicial vector spaces, means that all the β_n are expressible through the linear form β_1 :

$$\beta_n(a_0, a_1, \dots, a_n) = \beta_1(a_0 a_1 \cdots a_n).$$

The further conditions on β_1 express \mathfrak{G}_n -invariance of β_n for $n \geq 0$. Note that these conditions do not necessarily imply that the bilinear form β_2 is symmetric.

Definition 1.7.4. A $\Delta\mathfrak{G}$ -Frobenius algebra is a finite-dimensional unital associative algebra A equipped with a twisted \mathfrak{G}_0 -action and a $\Delta\mathfrak{G}$ -trace β such that the bilinear form β_2 is non-degenerate.

1.7.C. Planar examples

Here we illustrate the above construction for some of the planar $\Delta\mathfrak{G}$.

Cyclic case (Connes). Here $\Delta\mathfrak{G} = \Lambda$. Let A be an associative algebra. The graded vector space $C_\bullet(A)$ can be seen as a linearized, 1-object version of the cyclic nerve. The fact that $\text{NC}_\bullet\mathcal{C}$ is cyclic, corresponds to the fundamental observation of Connes that $C_\bullet(A)$ has a natural structure of a cyclic vector space. In this structure, the generator τ_n of $\mathbb{Z}/(n+1) = \text{Aut}\langle n \rangle$ acts by cyclic rotation

$$\tau_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

The concept of a $\Delta\mathfrak{G}$ -trace on A reduces, for $\Delta\mathfrak{G} = \Lambda$, to that of a linear functional $\beta_1 : A \rightarrow \mathbf{k}$ such that for each n the multilinear form $\beta_1(a_0 \cdots a_n)$ is cyclically symmetric. As well known, this is equivalent to requiring that $\beta_1(ab) = \beta_1(ba)$, so β_1 is the trace in the classical sense and β_2 is an invariant symmetric bilinear form on A . Further, the concept of a Λ -Frobenius algebra reduces to the classical one: a not necessarily commutative, associative unital \mathbf{k} -algebra with a trace β_1 such that $\beta_1(ab)$ is a nondegenerate bilinear form. It is classical that Frobenius algebras give rise to topological quantum field theories (TQFTs) on oriented surfaces, see [41] for systematic exposition. In this correspondence, commutative Frobenius algebras give rise to closed TQFTs defined on 2-dimensional oriented cobordisms without any additional data. Not necessarily commutative Frobenius algebra give rise to open TQFTs defined on oriented cobordisms together with a choice of marked points which are combinatorial representatives of punctures, cf. §3.1.

Dihedral case (Loday). Here $\Delta\mathfrak{G} = \Xi$. Let $(A, *)$ be a unital associative algebra with involution. As observed by Loday [46, §5.2.11] $C_\bullet(A)$ becomes a dihedral vector space, with ω_n acting by

$$\omega_n(a_0 \otimes \cdots \otimes a_n) = a_0^* \otimes a_n^* \otimes a_{n-1}^* \otimes \cdots \otimes a_2^* \otimes a_1^*.$$

A Ξ -trace on A is a linear functional $\beta_1 : A \rightarrow \mathbf{k}$ satisfying the identities

$$\beta_1(a^*) = \beta_1(a), \quad \beta_1(ab) = \beta_1(ba),$$

which imply that $\beta_1(ab) = \beta_1(a^*b^*)$, i.e., that the involution preserves the bilinear form β_2 (as well as all the higher β_n). Ξ -Frobenius algebras can be called “Frobenius algebras with involution”. They were used by Alexeevski and Natanzon [1] to construct TQFTs on unoriented surfaces.

Paracyclic and N -cyclic cases. Here $\Delta\mathfrak{G} = \Lambda_\infty$ or Λ_N . Suppose A is a unital associative algebra equipped with an automorphism F . We then have an invertible transformation τ_n on $C_n(A)$ given by

$$(1.7.5) \quad \tau_n(a_0, \dots, a_n) = (F^{-1}(a_n), a_0, a_1, \dots, a_{n-1}).$$

This makes $C_\bullet(A)$ into a paracyclic vector space. If $F^N = \text{Id}$ for some N , then $C_\bullet(A)$ is an N -cyclic vector space.

A Λ_∞ -trace on A is a linear functional $\beta_1 : A \rightarrow \mathbf{k}$ such that

$$\beta_1(F(a)) = \beta_1(a), \quad \beta_1(a \cdot b) = \beta_1(b \cdot F(a)).$$

In particular, the form $\beta_2(a, b) = \beta_1(ab)$ is not required to be symmetric, and F intertwines β_2 and β_2^t :

$$(1.7.6) \quad \beta_2(a, b) = \beta_2(b, F(a)).$$

Note that in the Frobenius case (A finite-dimensional and β_2 , considered as a morphism $A \rightarrow A^*$, is an isomorphism), F is defined by (1.7.6) uniquely: $F = \beta_2^{-1} \beta_2^t$. In this context F is called the *Nakayama automorphism* of β_2 . We get the following.

Proposition 1.7.7. *(a) A Λ_∞ -Frobenius algebra is the same as a finite-dimensional unital associative algebra A together with a linear functional $\beta_1 : A \rightarrow \mathbf{k}$ such that:*

(a1) $\beta_2(a, b) = \beta_1(ab)$ is a non-degenerate, not necessarily symmetric bilinear form on A .

(a2) The Nakayama automorphism F of β_2 is an algebra automorphism of A .

(b) A Λ_N -Frobenius algebra is a Λ_∞ -Frobenius algebra such that $F^N = \text{Id}$.

□

For $N = 2$, we obtain the concept studied under the name “Spin Frobenius algebras” in the work of Novak and Runkel [56] where such algebras are used to construct TQFTs on surfaces with 2-spin structure.

Quaternionic case (Dunn): Here $\Delta\mathfrak{G} = \nabla$. Let A be an associative algebra with an anti-automorphism J of order 4. Then the transformation

$$w_n(a_0 \otimes \dots \otimes a_n) = J^{-1}(a_0) \otimes J(a_n) \otimes \dots \otimes J(a_1)$$

together with τ_n defined as in (1.7.5) for $F = J^2$, make $C_\bullet(A)$ into a quaternionic object in the category of vector spaces.

A ∇ -trace on A is a linear functional $\beta : A \rightarrow \mathbf{k}$ such that $\beta_1(Ja) = \beta_1(a)$ and $\beta_2(a, b) = \beta_1(ab)$ satisfies the conditions:

$$\beta_2(a, b) = \beta_2(b, J^2a) = \beta_2(J^{-1}a, Jb).$$

The remaining planar cases can be analyzed in a similar way.

Remarks 1.7.8. (a) Let $\Delta\mathfrak{G}$ be any planar simplicial group with corresponding planar Lie group \mathbb{G} . We expect that the theory of $\Delta\mathfrak{G}$ -structured graphs developed in §4 can be used to generalize the results of [41, 1, 56] as follows: A $\Delta\mathfrak{G}$ -Frobenius algebra gives rise to a numerical invariant defined on \mathbb{G} -structured surfaces without boundary and to a TQFT on \mathbb{G} -structured 2-dimensional cobordisms (with marked points as in §3.1 allowed in both cases and with additional constraints of commutativity removing the dependence on marking points).

(b) Further, one can extend the concept of a $\Delta\mathfrak{G}$ -Frobenius algebra to the case of associative dg (or A_∞) algebras, and dg (or A_∞) categories also allowing the forms β_1 and β_2 to have some degree $d \neq 0$ as in [42]. The corresponding “ d -dimensional $\Delta\mathfrak{G}$ -Calabi-Yau algebras and categories” should then give cohomology classes on moduli spaces of marked \mathbb{G} -structured surfaces generalizing [16, 42].

We leave both these directions to future work.

2 Crossed simplicial groups and generalized orders

2.1 $\Delta\mathfrak{G}$ -Structured sets

The simplex category Δ , whose objects are the standard ordinals $[n]$, $n \geq 0$, can be embedded into a larger category $\mathbf{\Delta}$ with objects being all finite nonempty linearly ordered sets. While this embedding is an equivalence of categories, it is often more convenient to work with the larger category. In this section, we provide an analogous construction for any crossed simplicial group $\Delta\mathfrak{G}$. We introduce a larger category \mathcal{G} of $\Delta\mathfrak{G}$ -structured finite sets, fitting into a diagram of functors with vertical arrows being surjective on objects and horizontal arrows being equivalences of categories:

$$(2.1.1) \quad \begin{array}{ccc} \Delta\mathfrak{G} & \xrightarrow{\epsilon} & \mathcal{G} \\ \lambda \downarrow & & \downarrow \lambda_{\mathcal{G}} \\ \mathbf{N} & \longrightarrow & \mathbf{FSet} \end{array}$$

here \mathbf{FSet} is the category of all nonempty finite sets, $\mathbf{N} \subset \mathbf{FSet}$ is the full subcategory on the standard objects $\{0, 1, \dots, n\}$, and λ is the functor from Proposition 1.1.7.

2.1.A. Combinatorial model

If we assume that a category \mathcal{G} as in (2.1.1) is already constructed, then any object $I \in \mathcal{G}$ is isomorphic to a unique $[n] \in \Delta\mathfrak{G}$ and so $\text{Isom}_{\mathcal{G}}([n], I)$ is a right torsor over $\mathfrak{G}_n = \text{Isom}_{\Delta\mathfrak{G}}([n], [n])$. This leads to the following two definitions.

Definition 2.1.2. Let $\Delta\mathfrak{G}$ be a crossed simplicial group. A $\Delta\mathfrak{G}$ -structured set is a pair $(I, \mathcal{O}(I))$, where:

- (1) I is a nonempty finite set of some cardinality $n + 1$, $n \geq 0$.
- (2) $\mathcal{O}(I)$ is a right \mathfrak{G}_n -torsor together with a map

$$\rho : \mathcal{O}(I) \longrightarrow \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I)$$

which is equivariant with respect to the homomorphism $\lambda_n : \mathfrak{G}_n \rightarrow S_{n+1}$ from Proposition 1.1.7.

Thus ρ is a *reduction of structure group* inducing an isomorphism of S_{n+1} -torsors

$$\mathcal{O}(I) \times_{\mathfrak{G}_n} S_{n+1} \xrightarrow{\sim} \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I).$$

The datum of $\mathcal{O}(I)$ will be referred to as a $\Delta\mathfrak{G}$ -structure, or a $\Delta\mathfrak{G}$ -order, on the set I . Elements of $\mathcal{O}(I)$ will be called *structured frames* of I .

Definition 2.1.3. A morphism of $\Delta\mathfrak{G}$ -structured sets

$$\psi : (I', \mathcal{O}(I')) \longrightarrow (I, \mathcal{O}(I)), \quad |I'| = n' + 1, |I| = n + 1,$$

is a datum of morphisms

$$\{\psi_{f,f'} \in \text{Hom}_{\Delta\mathfrak{G}}([n'], [n]) \mid f' \in \mathcal{O}(I'), f \in \mathcal{O}(I)\},$$

satisfying, for every $g \in \mathfrak{G}_n$, $g' \in \mathfrak{G}_{n'}$, the equivariance condition

$$\psi_{fg, f'g'} = g^{-1} \circ \psi_{f,f'} \circ g'.$$

The composition of two morphisms

$$(I'', \mathcal{O}(I'')) \xrightarrow{\psi'} (I', \mathcal{O}(I')) \xrightarrow{\psi} (I, \mathcal{O}(I))$$

is defined by the formula

$$(\psi\psi')_{f,f''} = \psi_{f,f'} \circ \psi'_{f',f''}$$

where the right hand is independent on f' .

We denote by \mathcal{G} the category of $\Delta\mathfrak{G}$ -structured sets thus defined. The following is then obvious.

Proposition 2.1.4. (a) We have an equivalence of categories $\epsilon : \Delta\mathfrak{G} \rightarrow \mathcal{G}$ sending an object $[n]$ to the $\Delta\mathfrak{G}$ -structured set $(\{0, 1, \dots, n\}, \mathfrak{G}_n)$.

(b) For any object $(I, \mathcal{O}(I)) \in \mathcal{G}$, we have a canonical identification of \mathfrak{G}_n -torsors

$$\mathcal{O}(I) \longrightarrow \text{Isom}_{\mathcal{G}}(\epsilon[n], (I, \mathcal{O}(I)))$$

and of sets

$$I \longrightarrow \text{Hom}_{\mathcal{G}}(\epsilon[0], (I, \mathcal{O}(I)))/\mathfrak{G}_0.$$

(c) In particular, the correspondence $(I, \mathcal{O}(I)) \mapsto I$ gives a functor $\lambda_{\mathcal{G}} : \mathcal{G} \rightarrow \mathbf{FSet}$ extending $\lambda : \Delta\mathfrak{G} \rightarrow \mathbf{N}$. \square

In classical combinatorics, it is often important to speak about different orders of a specific type on a given set, without identifying isomorphic ones. For example, an N -element set has $N!$ different linear orders, all isomorphic to one another. The corresponding concept for $\Delta\mathfrak{G}$ -orders is as follows.

Definition 2.1.5. Let I be a nonempty finite set. The category \mathcal{G}/I of $\Delta\mathfrak{G}$ -orders on I has objects given by all $\Delta\mathfrak{G}$ -structures on I , and morphisms given by those morphisms $\psi : (I, \mathcal{O}) \rightarrow (I, \mathcal{O}')$ in \mathcal{G} which induce the identity map on the set I .

Clearly, \mathcal{G}/I is a groupoid. If $|I| = n + 1$ and $|\mathfrak{G}_n| < \infty$, then the number of isomorphism classes of objects of \mathcal{G}/I is

$$|\pi_0(\mathcal{G}/I)| = \frac{(n+1)!}{|\mathfrak{G}_n|} \cdot |\mathfrak{G}_n^0|, \quad \text{where } \mathfrak{G}_n^0 = \text{Ker}(\lambda_n).$$

This number can be understood as “the number of different $\Delta\mathfrak{G}$ -orders on I ”. Each such order, represented by an object of \mathcal{G}/I , has automorphism group isomorphic to \mathfrak{G}_n^0 .

Example 2.1.6. Consider the crossed simplicial group $\Delta\mathfrak{G} = \Xi$ given by the dihedral category. We have $\mathfrak{G}_0 = \mathfrak{G}_0^0 = \mathbb{Z}/2\mathbb{Z}$ so that, for a singleton set I , the category \mathcal{G}/I has a single isomorphism class of objects and each object has automorphism group $\mathbb{Z}/2\mathbb{Z}$. We have $\mathfrak{G}_1 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathfrak{G}_1^0 = \mathbb{Z}/2\mathbb{Z}$ so that, on a given set I of cardinality 2, we have one isomorphism class of objects in \mathcal{G}/I , each object having automorphism group $\mathbb{Z}/2\mathbb{Z}$. For $|I| = n + 1$ with $n \geq 2$, the objects of \mathcal{G}/I have trivial automorphism group, and there are $\frac{n!}{2}$ isomorphism classes.

Note that Definition 2.1.3 does not use any special features of $\Delta\mathfrak{G}$ except the functor $\lambda : \Delta\mathfrak{G} \rightarrow \mathbf{N}$. Using the unique factorization property of crossed simplicial groups, we can give the following reformulation.

Proposition 2.1.7. *The set $\text{Hom}_{\mathcal{G}}((I', \mathcal{O}(I')), (I, \mathcal{O}(I)))$ is in bijection with the set of pairs (ψ_*, ψ^*) , where $\psi_* : I' \rightarrow I$ and $\psi^* : \mathcal{O}(I) \rightarrow \mathcal{O}(I')$ are maps of sets such that:*

- (i) *For every structured frame f of I , the unique map $f^*\psi : \{0, 1, \dots, n'\} \rightarrow \{0, 1, \dots, n\}$ making the diagram*

$$\begin{array}{ccc} \{0, 1, \dots, n'\} & \xrightarrow{f^*\psi} & \{0, 1, \dots, n\} \\ \rho(\psi^*f) \downarrow & & \downarrow \rho(f) \\ I' & \xrightarrow{\psi_*} & I \end{array}$$

commutative is given by a morphism in Δ .

- (ii) *For each frame f of I and $g \in \mathfrak{G}_n$, we have*

$$\psi^*(f.g) = \psi^*(f).(f^*\psi)^*(g),$$

where $(f^\psi)^* : \mathfrak{G}_n \rightarrow \mathfrak{G}_{n'}$ is the map associated to $f^*\psi$ via the simplicial set \mathfrak{G} .*

Note that the diagram in (i) can be seen as an analog of the diagram (1.1.3) describing the canonical factorization of morphisms in $\Delta\mathfrak{G}$. The condition in (ii) mimics the first identity of crossed simplicial groups in Proposition 1.1.4(c).

Proof of Proposition 2.1.7: Let $\psi = (\psi_{f,f'})$ be a morphism in \mathcal{G} from $(I', \mathcal{O}(I'))$ to $(I, \mathcal{O}(I))$. The map $\psi_* : I' \rightarrow I$ is defined to be the value, on ψ , of the functor $\lambda_{\mathcal{G}}$. To define the

map $\psi^* : \mathcal{O}(I) \rightarrow \mathcal{O}(I')$ associated to ψ , we fix $f \in \mathcal{O}(I)$, choose an arbitrary $f' \in \mathcal{O}(I')$ and interpret them as vertical isomorphisms in the diagram

$$\begin{array}{ccc}
(I', \mathcal{O}(I')) & \xrightarrow{\psi} & (I, \mathcal{O}(I)) \\
f' \uparrow & & \uparrow f \\
\epsilon[n'] & \xrightarrow{\psi_{f,f'} = f^{-1} \circ \psi \circ f'} & \epsilon[n] \\
g' \uparrow & \nearrow \phi & \\
\epsilon[n'] & &
\end{array}$$

After this, we define ϕ and g' in the diagram via the canonical factorization $\psi_{f,f'} = \phi \circ g'$ with $\phi \in \Delta$, $g' \in \mathfrak{S}_{n'}$, and define $\psi^*(f) = f' \circ g'$. The result is independent on f' in virtue of the equivariance condition on the datum $\psi = (\psi_f, f')$. This defines the pair (ψ_*, ψ^*) associated to ψ . We leave the remaining verifications to the reader. \square

Example 2.1.8. A linear order \leq on a nonempty finite set I defines a canonical $\Delta\mathfrak{S}$ -order on I called the $\Delta\mathfrak{S}$ -closure of \leq . To define it, note that \leq can be identified with the unique monotone bijection $\sigma : \{0, 1, \dots, n\} \rightarrow I$, where $|I| = n + 1$. We then define the \mathfrak{S}_n -torsor \mathcal{O}_σ to be \mathfrak{S}_n , and the map ρ to be the composition

$$\mathfrak{S}_n \xrightarrow{\lambda_n} S_{n+1} = \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, \{0, 1, \dots, n\}) \xrightarrow{\sigma_*} \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I).$$

Denoting by Δ the category of all nonempty finite ordinals, we can view the $\Delta\mathfrak{S}$ -closure as a functor $\mathbf{i} : \Delta \rightarrow \mathcal{G}$ extending $i : \Delta \rightarrow \Delta\mathfrak{S}$.

Proposition 2.1.9. (a) For two linear orders on I given by bijections σ_1, σ_2 , we have

$$\text{Hom}_{\mathcal{G}/I}((I, \mathcal{O}_{\sigma_1}), (I, \mathcal{O}_{\sigma_2})) \cong \{g \in \mathfrak{S}_n \mid \lambda_n(g) = \sigma_1^{-1} \sigma_2\},$$

and the composition of morphisms corresponds to the multiplication in \mathfrak{S}_n .

(b) Every object $(I, \mathcal{O}(I))$ of \mathcal{G}/I is isomorphic to an object of the form (I, \mathcal{O}_σ) , so the category with objects (I, \mathcal{O}_σ) and morphisms defined by the right hand side of the identification in (a), is equivalent to \mathcal{G}/I .

Proof: (a) is straightforward from the definitions. To prove (b), note that any $f \in \mathcal{O}(I)$ gives rise to an isomorphism $\psi^f : [n] \rightarrow (I, \mathcal{O}(I))$ in \mathcal{G} which gives a bijection $\sigma = \lambda(\psi^f)$ from $\{0, 1, \dots, n\}$ to I . We then see easily that $(I, \mathcal{O}(I))$ is isomorphic to (I, \mathcal{O}_σ) by an isomorphism identical on I . \square

Note that a Δ -structured set is a finite nonempty linearly ordered set. Any subset of a linearly ordered set admits a canonical induced order. The following proposition shows that this fundamental property generalizes to $\Delta\mathfrak{S}$ -structured sets.

Proposition 2.1.10. *Let $u : I' \hookrightarrow I$ be an embedding of nonempty finite sets. Then there is a functor*

$$\text{Res}_{I'}^I : \mathcal{G}/I \longrightarrow \mathcal{G}/I'$$

called restriction of $\Delta\mathfrak{G}$ -structures from I to I' , with the following properties:

(a) *For each $\Delta\mathfrak{G}$ -structure $\mathcal{O}(I)$ on I the map u extends to a canonical morphism in \mathcal{G}*

$$\psi^u : \text{Res}_{I'}^I(I, \mathcal{O}(I)) \longrightarrow (I, \mathcal{O}(I)).$$

(b) *For a triple of composable embeddings $I'' \hookrightarrow I' \hookrightarrow I$ we have a canonical isomorphism*

$$\text{Res}_{I''}^I \simeq \text{Res}_{I''}^{I'} \circ \text{Res}_{I'}^I.$$

Proof. We use the model of \mathcal{G}/I from Proposition 2.1.9. Given a total order \leq on I , it restricts to a total order on I' , and so we have a diagram

$$(2.1.11) \quad \begin{array}{ccc} \{0, 1, \dots, n'\} & \xrightarrow{\phi} & \{0, 1, \dots, n\} \\ \sigma' \downarrow & & \downarrow \sigma \\ I' & \xrightarrow{u} & I, \end{array}$$

in which ψ, u are monotone embeddings, and σ, σ' are monotone bijections. We define on objects $\text{Res}_{I'}^I(I, \mathcal{O}_\sigma) = (I', \mathcal{O}_{\sigma'})$.

Further, let $g \in \mathfrak{G}_n$ be such that $\lambda_n(g) = \sigma_1^{-1}\sigma_2$, where $\sigma_1, \sigma_2 : \{0, 1, \dots, n\} \rightarrow I$ are the bijections corresponding to two total orders \leq^1, \leq^2 on I . Let $\sigma'_1, \sigma'_2 : \{0, 1, \dots, n'\} \rightarrow I'$ be the bijections corresponding to the restrictions of \leq^1, \leq^2 to I' . We then have two commutative diagrams as in (2.1.11), whose arrows we denote $u, \sigma'_\nu, \sigma_\nu, \phi_\nu, \nu = 1, 2$.

We claim, first of all, that $\phi_2 = g^*\phi_1$. Indeed, consider the canonical diagram in $\Delta\mathfrak{G}$

$$\begin{array}{ccc} [n'] & \xrightarrow{g^*\phi_1} & [n] \\ \phi_1^*g \downarrow & & \downarrow g \\ [n'] & \xrightarrow{\phi_1} & [n] \end{array}$$

defining $g^*\phi_1$ and apply the functor λ to it. Because of the assumptions on g , the resulting diagram in \mathbf{FSet} is

$$(2.1.12) \quad \begin{array}{ccc} \{0, 1, \dots, n'\} & \xrightarrow{\phi_2} & \{0, 1, \dots, n\} \\ \lambda_n(\phi_1^*g) \downarrow & & \downarrow \lambda_n(g) = \sigma_1^{-1}\sigma_2 \\ \{0, 1, \dots, n'\} & \xrightarrow{\phi_1} & \{0, 1, \dots, n\}, \end{array}$$

whence $\phi_2 = g^*\phi_1$.

We further claim that $\lambda_n(\phi_1^*(g)) = (\sigma'_1)^{-1}\sigma'_2$. Indeed, in virtue of the injectivity of ϕ_1, ϕ_2 and bijectivity of $\lambda_n(g)$, there can be at most one map making (2.1.12) commutative, and both sides of the proposed equality do. So we define $\text{Res}_{I'}^I$ on morphisms by sending g to $\phi_1^*(g)$. To see that $\text{Res}_{I'}^I$ commutes with composition of morphisms, we consider three orders \leq^ν on I , $\nu = 1, 2, 3$ with three corresponding bijections σ_ν and monotone maps ϕ_ν . If now $g_1 = \sigma_1^{-1}\sigma_2$, $g_2 = \sigma_2^{-1}\sigma_3$, then $\text{Res}_{I'}^I(g_1g_2) = \phi_1^{-1}(g_1g_2)$, while

$$\text{Res}_{I'}^I(g_1) \circ \text{Res}_{I'}^I(g_2) = \phi_1^*(g_1)\phi_2^*(g_2) = \phi_1^*(g_1) \circ ((g_1^*\phi_1)^*(g_2)),$$

which equals $\phi^*(g_1g_2)$ by the first equality in Proposition 1.1.4(c). We leave the remaining verifications to the reader. \square

Remark 2.1.13. Given a structure torsor

$$\mathcal{O}(I) \xrightarrow{\rho} \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I)$$

for I , we can directly construct the “restricted torsor” $\mathcal{O}(I')$ for $\text{Res}_{I'}^I(I, \mathcal{O}(I))$ and the canonical morphism

$$\psi = (\psi_* = u, \psi^*) : (I', \mathcal{O}(I')) \rightarrow (I, \mathcal{O}(I))$$

as follows. Choose some $f \in \mathcal{O}(I)$, so that $\sigma = \lambda_{\mathcal{G}}(f)$ is a bijection defining a total order on I . Let $\phi : [n'] \rightarrow [n]$ be the monotone map in (2.1.11). To emphasize the dependence of ϕ on f , we denote it as $\phi = f^*\psi$ (where ψ stands for the morphism we are constructing). For $g, h \in \mathfrak{G}_n$ put $f.g \sim f.h$ whenever $(f^*u)^*(g) = (f^*u)^*(h)$ in $\mathfrak{G}_{n'}$. It is immediate that this is an equivalence relation, independent on the choice of f . We put $\mathcal{O}(I') = \mathcal{O}(I)/\sim$ and define $\psi^* : \mathcal{O}(I) \rightarrow \mathcal{O}(I')$ to be the quotient map. The composite

$$\mathcal{O}(I) \longrightarrow \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I) \longrightarrow \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n'\}, I)$$

factors through ψ^* providing a map $\rho_{I'} : \mathcal{O}(I') \rightarrow \text{Isom}(\{0, 1, \dots, n'\}, I)$. There exists a unique $\mathfrak{G}_{n'}$ -action on $\mathcal{O}(I')$ such that, for every $f \in \mathcal{O}(I)$ and $g \in \mathfrak{G}_n$, we have

$$\psi^*(f.g) = \psi^*(f).((f^*\psi)^*g).$$

It is now readily verified that this \mathfrak{G}_m -action on $\mathcal{O}(I)$ is simply transitive, and the maps $\psi_* = u$ and ψ^* satisfy the conditions (i) and (ii) of Proposition 2.1.7.

2.1.B. Topological model

Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group with corresponding planar Lie group $p : G \rightarrow O(2)$. In this section, we define a topological model for $\Delta\mathfrak{G}$ based on the planar homeomorphism group

$$p_{\text{Homeo}} : \text{Homeo}^G(S^1) \longrightarrow \text{Homeo}(S^1)$$

corresponding to G as introduced in §1.5.B.

By a *circle* we mean a topological space homeomorphic to the standard circle S^1 . By a *marked circle* we mean a pair (C, J) where C is a circle and $J \subset C$ is a nonempty closed subset homeomorphic to a finite disjoint union of copies of the interval $[0, 1]$. For two marked circles (C, J) and (C', J') , we denote

$$\text{Homeo}((C, J), (C', J')) = \{\phi \in \text{Homeo}(C, C') \mid \phi(J) \subset J'\}.$$

It has a natural topology as a closed subspace of $\text{Homeo}(C, C')$.

Proposition 2.1.14. *Each connected component of $\text{Homeo}((C, J), (C', J'))$ is contractible.*

Proof. Follows from the fact that the group of orientation preserving self-homeomorphisms of an interval is contractible. \square

By a G -structured circle¹ we mean a pair (C, ρ) where C is a circle, and $\rho : F \rightarrow \text{Homeo}(S^1, C)$ is a reduction of the structure group along p_{Homeo} . Thus, F is a $\text{Homeo}^G(S^1)$ -torsor and ρ is p_{Homeo} -equivariant. We will often denote a G -structured circle simply by C , assuming ρ to be given.

For G -structured circles (C, ρ) and (C', ρ') , we define $\text{Homeo}^G((C, \rho), (C', \rho'))$ to be the set of pairs $(\phi, \tilde{\phi})$, where $\phi \in \text{Homeo}(C, C')$ and $\tilde{\phi} : F \rightarrow F'$ is a $\text{Homeo}^G(S^1)$ -equivariant homeomorphism such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\phi}} & F' \\ \rho \downarrow & & \downarrow \rho' \\ \text{Homeo}(S^1, C) & \xrightarrow{\phi} & \text{Homeo}(S^1, C') \end{array}$$

is commutative. Note that the set $\text{Homeo}^G((C, \rho), (C', \rho'))$ has a natural topology with respect to which it is homeomorphic to $\text{Homeo}^G(S^1)$.

We will consider circles which are both structured and marked, denoting them by (C, J) where $C = (C, \rho)$ is a G -structured circle. For G -structured marked circles (C, J) and (C', J') , we denote by $\text{Homeo}^G((C, J), (C', J'))$ the closed subspace in $\text{Homeo}^G(C, C')$ formed by $(\phi, \tilde{\phi})$ such that $\phi \in \text{Homeo}((C, J), (C', J'))$.

Definition 2.1.15. We define the category \mathcal{C}_G with objects given by G -structured marked circles and morphisms

$$\text{Hom}_{\mathcal{C}_G}((C, J), (C', J')) = \pi_0 \text{Homeo}^G((C, J), (C', J')).$$

We have the main result of this section:

¹A more precise term would be “ $\text{Homeo}^G(S^1)$ -structured circle”.

Theorem 2.1.16. *Let $p : G \rightarrow O(2)$ be a planar Lie group with corresponding crossed simplicial group $\Delta\mathfrak{G}$. Then the functor*

$$\lambda_{\mathcal{C}_G} : \mathcal{C}_G \longrightarrow \mathbf{FSet}, \quad (C, J) \mapsto \pi_0(J)$$

canonically factors as in

$$\begin{array}{ccc} \mathcal{C}_G & \xrightarrow[\simeq]{\pi} & \mathcal{G} \\ & \searrow \lambda_{\mathcal{C}_G} & \swarrow \lambda_{\mathcal{G}} \\ & \mathbf{FSet} & \end{array}$$

where \mathcal{G} denotes the category of $\Delta\mathfrak{G}$ -structured sets and $\pi : \mathcal{C}_G \rightarrow \mathcal{G}$ is an equivalence of categories. In particular, the category \mathcal{C}_G is equivalent to $\Delta\mathfrak{G}$.

Proof. Let (C, J) be a marked G -structured circle with reduction $\rho : F \rightarrow \text{Homeo}(S^1, C)$ of structure group along p_{Homeo} . We claim that the association

$$(C, J) \mapsto (\pi_0(J), \rho_{\pi_0(J)}),$$

where $(\pi_0(J), \rho_{\pi_0(J)})$ denotes the $\Delta\mathfrak{G}$ -structured set constructed in Lemma 2.1.17 below, extends to an equivalence of categories $\pi : \mathcal{C}_G \rightarrow \Delta\mathfrak{G}$. Given a morphism $(\varphi, \tilde{\varphi}) : (C, J) \rightarrow (C', J')$ of marked G -structured circles, we have to construct a morphism of the $\Delta\mathfrak{G}$ -structured sets $(\pi_0(J), \rho_{\pi_0(J)})$ and $(\pi_0(J'), \rho_{\pi_0(J')})$. Clearly, we have an induced map $\psi : \pi_0(J) \rightarrow \pi_0(J')$ of underlying sets. We now define the pullback map $\psi^* : \pi_0(F_{J'}) \rightarrow \pi_0(F_J)$ of structured frames. Let $f \in F_{J'}$ represent a structured frame in $\pi_0(F_{J'})$. Its image under ρ defines a homeomorphism $\rho(f) : \text{Homeo}(S^1, C')$ which maps $[n]$ to J' . We choose a point in the unique component of $C' \setminus J'$ bounded by $\rho(f)(n)$ and $\rho(f)(0)$, and denote its preimage in C under φ by p . Note that p lies in $C \setminus J$. Consider the homeomorphism $\varphi^{-1} \circ \rho(f)$ in $\text{Homeo}(S^1, C)$. Now we choose a path $\alpha : [0, 1] \rightarrow \text{Homeo}(S^1, C)$ starting at $\varphi^{-1} \circ \rho(f)$, ending at a homeomorphism which maps $[m]$ to J , such that, for all $t \in [0, 1]$, $\alpha(t)(0) \neq p$. It is easy to see that the space of such paths is contractible. Since $F \rightarrow \text{Homeo}(S^1, C)$ is a covering, there is a unique lift of α to F satisfying $\alpha(0) = \tilde{\varphi}$. We define $\psi^*(f)$ to be the connected component of $\alpha(1)$ in the space F_J . The pair (ψ_*, ψ^*) defines a morphism in \mathcal{G} by Proposition 2.1.7. It is straightforward to show that this construction is functorial. To see that it provides an equivalence of categories it suffices to verify the unique factorization property for the full subcategory of \mathcal{C}_G generated by a set of standard objects $\{(S^1, [n])\}$ which is easily done. \square

Lemma 2.1.17. *Let $n \geq 0$ and consider the standard circle S^1 equipped with the standard subset $[n] \subset S^1$ of $(n+1)$ st roots of unity.*

- (1) *Consider the subgroup $\text{Homeo}(S^1, [n]) \subset \text{Homeo}(S^1)$ of homeomorphisms preserving which preserve the standard subset $[n]$ of $(n+1)$ st roots of unity. Let H_n be the pullback*

$$\begin{array}{ccc} H_n & \longrightarrow & \text{Homeo}^G(S^1) \\ \downarrow & & \downarrow p_{\text{Homeo}} \\ \text{Homeo}(S^1, [n]) & \longrightarrow & \text{Homeo}(S^1). \end{array}$$

Then each connected component of H_n is contractible and there is a canonical group isomorphism $\pi_0(H_n) \cong \mathfrak{G}_0$.

- (2) Let (C, J) be a G -structured marked circle with reduction $\rho : F \rightarrow \text{Homeo}(S^1, C)$ of structure group along p_{Homeo} . Consider the subset $\text{Homeo}((S^1, [n]), (C, J)) \subset \text{Homeo}(S^1, C)$ consisting of homeomorphisms which map $[n]$ into J and let $\rho_J : F_J \rightarrow \text{Homeo}((S^1, [n]), (C, J))$ denote the restriction of ρ . Then the set $\pi_0(F_J)$ is a $\pi_0(H_n)$ -torsor, equipped with a natural equivariant map

$$\rho_{\pi_0(J)} : \pi_0(F_J) \rightarrow \pi_0(J)$$

making $(\pi_0(J), \rho_{\pi_0(J)})$ a $\Delta\mathfrak{G}$ -structured set.

Proof. (1) As argued in Proposition 2.1.14, each component of the subgroup $\text{Homeo}(S^1, [n])$ is contractible. Therefore, the map p_{Homeo} restricts to a covering $H_n \rightarrow \text{Homeo}(S^1, [n])$ which is topologically trivial so that each connected component of H_n must be contractible. We have

$$\pi_0 \text{Homeo}(S^1, [n]) \cong D_{n+1} \cong \text{Aut}_{\Xi}([n])$$

where Ξ denotes the dihedral category of Example 1.4.2. The group $\pi_0(H_n)$ can be identified with the preimage $p^{-1}(D_{n+1})$ under $p : G \rightarrow O(2)$. By Theorem 1.5.6(b2), this preimage is canonically identified with \mathfrak{G}_n .

- (2) Passing to connected components, we obtain a sequence of maps

$$\pi_0(F_J) \xrightarrow{\pi_0(\rho_J)} \pi_0(\text{Homeo}((S^1, [n]), (C, J))) \longrightarrow \text{Isom}_{\mathbf{FSet}}([n], \pi_0(J))$$

whose composite is defined to be $\rho_{\pi_0(J)}$. It is apparent that $(\pi_0(J), \rho_{\pi_0(J)})$ is a $\Delta\mathfrak{G}$ -structured set. \square

Corollary 2.1.18. *Let C be a G -structured circle. Then, for any nonempty finite subset $I \subset C$, both I and $\pi_0(C - I)$ acquire canonical $\Delta\mathfrak{G}$ -structures. These structures are natural with respect to homeomorphisms of G -structured circles.*

The following fact generalizes the result of Drinfeld [19, §3].

Proposition 2.1.19. *Let $R : \text{Set}_{\Delta\mathfrak{G}} \rightarrow \mathcal{T}\text{op}$ be the geometric realization functor: $R(X) = |i^*X|$, where $i : \Delta \rightarrow \Delta\mathfrak{G}$ is the embedding. The group $\text{Homeo}^G(S^1)$ acts on R by natural isomorphisms. In particular, it acts on each $|i^*X|$ by homeomorphisms.*

It seems plausible that the group $\text{Homeo}^G(S^1)$ is in fact identified with $\text{Aut}(R)$ which would provide its natural construction in terms of $\Delta\mathfrak{G}$.

Proof: The argument is similar to that of [19]. That is, let Δ be the category of all nonempty finite ordinals, and $\mathbf{i} : \Delta \rightarrow \mathcal{G}$ be the functor of $\Delta\mathfrak{G}$ -closure, extending i (Example 2.1.8). A simplicial set Y can be seen as a contravariant functor $Y : \Delta \rightarrow \text{Set}$ and, as pointed out in Formula (1.1) of [19],

$$|Y| = \varinjlim_{F \subset [0,1]} Y(\pi_0([0,1] - F)),$$

where F runs over finite subsets of $[0, 1]$, and $\pi_0([0, 1] - F)$ is equipped by the total order induced by that on $[0, 1]$.

Now, if X is a $\Delta\mathfrak{G}$ -set, we can view X as a contravariant functor $\mathcal{G} \rightarrow \mathcal{S}et$ and identify

$$\begin{aligned} |i^* X| &= \varinjlim_{F \subset [0, 1]} (i^* X)(\pi_0([0, 1] - F)) = \varinjlim_{F \subset [0, 1]} X(i(\pi_0([0, 1] - F))) = \\ &= \varinjlim_{1 \in F \subset S^1} X(i(\pi_0(S^1 - F))) = \varinjlim_{F \subset S^1} X(i(\pi_0(S^1 - F))). \end{aligned}$$

Here we view S^1 as obtained from $[0, 1]$ by identifying 0 and 1 into one point $1 \in S^1$ and as equipped with the standard $\text{Homeo}^G(S^1)$ -structure. Each set $\pi_0(S^1 - F)$ is then equipped with a $\Delta\mathfrak{G}$ -structure by Corollary 2.1.18. Now, the last colimit is manifestly acted upon by $\text{Homeo}^G(S^1)$. \square

2.1.C. Interstice duality

The topological model \mathcal{C}_G allows for an immediate and unified proof of the following result which is well-known for cyclic (cf. [13, 19]), dihedral, and quaternionic categories (cf. [20]).

Corollary 2.1.20. *Any planar crossed simplicial group is self-dual.*

Proof. We define the functor $(-)^{\vee} : \mathcal{C}_G \rightarrow \mathcal{C}_G^{\text{op}}$ by putting

$$(C, J)^{\vee} := (C, \overline{C \setminus J}), \quad (\varphi, \tilde{\varphi})^{\vee} := (\varphi^{-1}, \tilde{\varphi}^{-1})$$

on objects and morphisms, respectively. Since the square of this functor equals the identity functor, it must be an equivalence.

Since $\Delta\mathfrak{G}$ is equivalent to \mathcal{C}_G , we conclude that there exists a duality functor $(-)^{\vee}_{\Delta\mathfrak{G}} : \Delta\mathfrak{G} \rightarrow \Delta\mathfrak{G}^{\text{op}}$ with $[n]^{\vee} = [n]$. The choice of values for this functor on morphisms, however, is not canonical. \square

The connected components of $\overline{C \setminus J}$ can be regarded as “interstices” positioned between adjacent connected components of J .

Remarks 2.1.21. (a) If we fix isomorphisms $[n]^{\vee} \simeq [n]$ for all n (i.e., fix a functor $(-)^{\vee}_{\Delta\mathfrak{G}}$), we get identifications of sets

$$\begin{aligned} \mathfrak{G}_n &\stackrel{\text{C.F.}}{=} \text{Hom}_{\Delta\mathfrak{G}}([n], [0]) \simeq \text{Hom}_{\Delta\mathfrak{G}}([0]^{\vee}, [n]^{\vee}) \simeq \\ &\simeq \text{Hom}_{\Delta\mathfrak{G}}([0], [n]) \stackrel{\text{C.F.}}{=} \mathfrak{G}_0 \times \text{Hom}_{\Delta\mathfrak{G}}([0], [n]), \end{aligned}$$

where “C.F.” stands for the canonical factorization of morphisms in a crossed simplicial group. These identifications can be interpreted by saying that \mathfrak{G}_n “grows linearly with n ” (with “coefficient of proportionality” being \mathfrak{G}_0). Cf. the discussion of three growth types of crossed simplicial groups in §1.2.

(b) The isomorphism $[n]^{\vee} \simeq [n]$ also implies that the \mathfrak{G}_n -action on $\text{Hom}_{\Delta\mathfrak{G}}([n], [n])$ is simply transitive.

Let us also give a purely combinatorial description of duality, in terms of the category \mathcal{G} of $\Delta\mathfrak{G}$ -structured sets. Note that for any object $(I, \mathcal{O}(I)) \in \Delta\mathfrak{G}$ we have canonical identifications

$$I = \text{Hom}_{\mathcal{G}}([0], I)/\mathfrak{G}_0, \quad \mathcal{O}(I) = \text{Hom}_{\mathcal{G}}(I, [0]).$$

Proposition 2.1.22. *Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group. Then:*

(1) *For any $(I, \mathcal{O}(I)) \in \mathcal{G}$ the pair*

$$(I^\vee, \mathcal{O}(I^\vee)) := (\text{Hom}_{\mathcal{G}}(I, [0])/\mathfrak{G}_0, \text{Hom}_{\mathcal{G}}([0], I))$$

is canonically a $\Delta\mathfrak{G}$ -structured set.

(2) *The association $I \mapsto I^\vee$ extends to a functor $\mathcal{G} \rightarrow \mathcal{G}^{\text{op}}$ which is an equivalence of categories.*

Proof. To show (1), we have to exhibit a canonical map

$$\rho^\vee : \mathcal{O}(I^\vee) \longrightarrow \text{Isom}_{\mathbf{FSet}}(\{0, 1, \dots, n\}, I^\vee)$$

which is a reduction of structure group with respect to $\mathfrak{G}_n \rightarrow S_{n+1}$. As explained above, we have a canonical bijection $I^\vee \cong \mathcal{O}(I)/\mathfrak{G}_0$. Let $i \in \mathcal{O}(I^\vee)$, corresponding to a morphism $i : \{0\} \rightarrow I$ in \mathcal{G} with corresponding pullback map $i^* : \mathcal{O}(I) \rightarrow \mathfrak{G}_0$. We define $\mathcal{O}(I)^i$ to be the preimage of 1 under i^* . The composite of the canonical maps

$$\mathcal{O}(I)^i \longrightarrow \mathcal{O}(I) \longrightarrow \mathcal{O}(I)/\mathfrak{G}_0$$

is a bijection of sets. Further, we have a canonical map

$$\mathcal{O}(I)^i \longrightarrow \{0, 1, \dots, n\}, \quad f \mapsto f^*i.$$

It follows from Remark 2.1.21(b) that this map is a bijection. Combining both bijections, we obtain the desired bijection $\{0, 1, \dots, n\} \rightarrow \mathcal{O}(I)/\mathfrak{G}_0$. This construction defines the map ρ^\vee .

To show (2), let $\psi : (I, \mathcal{O}(I)) \rightarrow (J, \mathcal{O}(J))$ be a morphism in \mathcal{G} . Let us construct $\psi^\vee : (J^\vee, \mathcal{O}(J^\vee)) \rightarrow (I^\vee, \mathcal{O}(I^\vee))$ in terms of the corresponding pair $(\psi_*^\vee, \psi^{\vee*})$, as in Proposition 2.1.7. We take $\psi_*^\vee : J^\vee \rightarrow I^\vee$ to be induced by the contravariant functoriality of $\text{Hom}_{\mathcal{G}}(-, [0])$, and $\psi^{\vee*} : \mathcal{O}(I^\vee) \rightarrow \mathcal{O}(J^\vee)$ to be induced by the covariant functoriality of $\text{Hom}_{\mathcal{G}}([0], -)$. The verification of required properties follows by comparison with the topological description. \square

In the subsequent sections, we explicitly analyze the category \mathcal{G} of $\Delta\mathfrak{G}$ -structured sets for various examples of planar crossed simplicial groups.

2.2 Cyclic orders

Let $\Delta\mathfrak{G} = \Lambda$ be Connes' cyclic category. A Λ -structure on a set J can then be interpreted as a *cyclic order*.

2.2.A. Finite total cyclic orders

Definition 2.2.1. A *(total) cyclic order* on a finite set J of cardinality n is the choice of a simply transitive action of the group $\mathbb{Z}/n\mathbb{Z}$. Note that a cyclic order on the set J is uniquely determined by specifying for every element j its *successor* $j + 1$.

To avoid confusion, we refer to the usual concept of total order as *linear order*.

Examples 2.2.2. (1) Any set of cardinality ≤ 2 has a unique cyclic order. A cyclic order on a set J of cardinality 3 is the same as an orientation of the triangle $|\Delta^J|$.

- (2) Every subset of a cyclically ordered set J has a canonical cyclic order.
- (3) Any finite linearly ordered set J carries a canonical cyclic order where the successor of each nonmaximal element of J is its successor with respect to the linear order and the minimal element is the successor of the maximal element. We say that this cyclic order is the *cyclic closure* of the linear order on J .
- (4) More generally, given a cyclically ordered finite set K and a map of finite sets $f : J \rightarrow K$ where each fiber of f is equipped with a linear order, we can define the *lexicographic* cyclic order on J as follows. First note, that the set of fibers of f is canonically identified with K and hence carries a cyclic order. Therefore, every fiber has a successor fiber. Now the cyclic order on J is given as follows: For every element j which is nonmaximal in its fiber, we define the successor $j + 1$ to be the successor with respect to the specified linear order of the fiber. If j is maximal in its fiber, we define its successor to be the minimal element of the successor fiber. Here the convention is to skip all empty fibers. The previous example is then given by the case when K is a point, equipped with its unique cyclic order.

Definition 2.2.3. A morphism $f : J \rightarrow K$ of finite cyclically ordered sets is a map $f : J \rightarrow K$ of underlying sets equipped with the choice of a linear order on every fiber such that the cyclic order on J is the induced lexicographic order. The resulting category $\mathbf{\Lambda}$ given by finite nonempty cyclically ordered sets and their morphisms is called the *large cyclic category*.

Remarks 2.2.4. (a) Restricting to the standard cyclic ordinals $\langle n \rangle$, this gives a purely combinatorial description of morphisms in Connes' category $\mathbf{\Lambda}$.

(b) We have a natural faithful embedding $\mathbf{\Delta} \subset \mathbf{\Lambda}$ given by associating to a linearly ordered set J its cyclic closure. This extends the embedding $\mathbf{\Delta} \subset \mathbf{\Lambda}$.

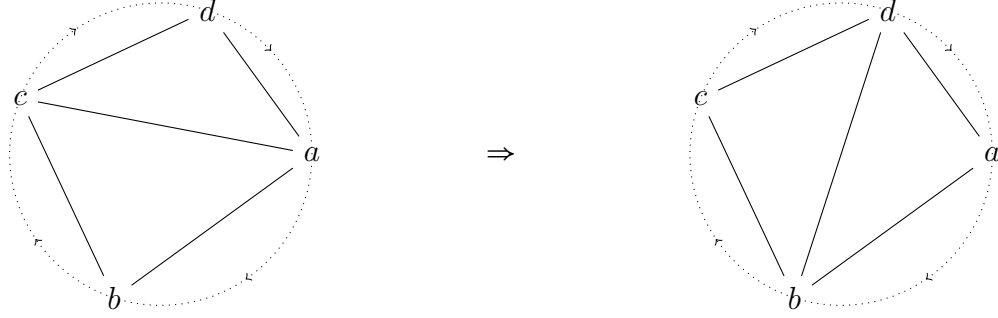
2.2.B. General cyclic orders

While the concept of cyclic order introduced above is sufficient for the main purposes of this work, it is not fully satisfactory in that it does not cover the prototypical example, generating all the finite cyclic orders: *the circle S^1 itself*. Indeed, S^1 is an infinite set.

The most flexible definition of a linear order is given by a binary relation on a set. It is therefore natural to define cyclic orders in a similar style. The following definition is a slight modification of those given by Huntington and Novák [34, 55].

Definition 2.2.5. Let J be a set. A *partial cyclic order* on J is a ternary relation $\lambda \subset J^3$ with the following properties:

- (C1) (reflexivity) We have $(a, a, a) \in \lambda$ for any $a \in J$.
- (C2) (antisymmetry) If $(a, b, c) \in \lambda$ and $(b, a, c) \in \lambda$, then $|\{a, b, c\}| \leq 2$.
- (C3) (transitivity) Let $a, b, c, d \in J$ be distinct. If $(a, b, c) \in \lambda$ and $(a, c, d) \in \lambda$, then $(a, b, d) \in \lambda$ and $(b, c, d) \in \lambda$:



- (C4) (cyclic symmetry) If $(a, b, c) \in \lambda$, then $(b, c, a) \in \lambda$.
- (C5) (2-cycle axiom) If $(a, b, c) \in \lambda$, then $(a, a, c) \in \lambda$.

A *total cyclic order* is a partial cyclic order satisfying the additional condition:

- (C6) (comparability) For any a, b, c we have either $(a, b, c) \in \lambda$, or $(b, a, c) \in \lambda$.

Note the formal similarity of (C3) and the Pachner move (exchange of two triangulations of a 4-gon). Note also that the implication in (C3) is in fact an equivalence, as we can apply it twice.

Examples 2.2.6. (1) Let C be a circle. Then $C^3_\# = C^3 - \bigcup_{i \neq j} \{x_i = x_j\}$ has two connected components which are in bijection with orientations of C . The “anti-clockwise” component corresponding to a choice of orientation contains triples (x_1, x_2, x_3) with x_i following very closely each other in the orientation direction. The closure λ of this component is a total cyclic order on C . In particular, the standard oriented circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ has a canonical cyclic order.

This implies that a total cyclic order on a finite set J can be defined as an homeomorphism class of embeddings of J into oriented circles. That is, two such embeddings

$u : J \rightarrow C$ and $u' : J \rightarrow C'$ are identified iff there is an orientation preserving homeomorphism $\phi : C \rightarrow C'$ such that $\phi u = u'$. It is now easy to verify that a total cyclic order on a finite set as defined in Definition 2.2.5 is the same as a total cyclic order in the sense of Definition 2.2.1.

- (2) Let \leq be a partial linear order on S . Its *cyclic closure* is the ternary relation $\lambda = \lambda_{\leq} \subset S^3$ obtained as the closure of the set $\{x_1 \leq x_2 \leq x_3\}$ under the action of $\mathbb{Z}/3$ by cyclic rotations. It is straightforward to see that λ is a partial cyclic order, total if \leq is a total linear order, cf. [55, Th. 3.5].

This implies that total cyclic order on a finite set J with $|J| = n$ is the same as a class of bijections $J \rightarrow \{1, 2, \dots, n\}$ under the action of \mathbb{Z}/n , each bijection (i.e., a total order) giving a cyclic order by cyclic closure.

- (3) Every set J has the *discrete partial cyclic order* consisting only of triples (a, a, a) , $a \in J$. If $|J| = 1$, this is the only partial cyclic order on J . If $|J| = 2$, then J has two partial cyclic orders: one discrete and the other is the total cyclic order, formed by all triples $(a, b, c) \in J^3$. Note that Novák's setting [55] allows only one partial cyclic order on a 2-element set, the discrete one.
- (4) Generalizing (1), one can see that the set of points of any “abstract circle” in the sense of Moerdijk [51] has a total cyclic order.

Example 2.2.7. The real projective line $\mathbb{R}P^1$, being a circle, has a cyclic order. Such an order is determined by a choice of orientation of \mathbb{R}^2 . More generally, let $F_n = \text{Flag}(1, 2, \dots, n; \mathbb{R}^n)$ be the space of complete flags in \mathbb{R}^n . For example, $F_2 = \mathbb{R}P^1$. One of the main results of [26] (theory of positive configurations of flags) says, in our terminology, that F_n has a natural partial cyclic order λ , determined by a choice of orientation of \mathbb{R}^n . Explicitly, a closed smooth curve $C \subset \mathbb{R}P^{n-1}$ is called *convex*, if every hyperplane intersects C in at most $n - 1$ points. In particular, such C does not lie in a hyperplane. Taking the osculating flags to C at all its points, we obtain an embedding $\gamma_C : C \rightarrow F_n$. The 2-element set of orientations of any convex curve $C \subset \mathbb{R}P^{n-1}$ is canonically identified with the 2-element set of orientations of \mathbb{R}^n , by a version of the Frenet frame construction (this is the old-fashioned definition of the orientation of the space in terms of “left and right screws”). Thus choosing an orientation of \mathbb{R}^n , we equip any convex C with a total cyclic order λ_C . We then say that $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) \in F_n^3$ lies in λ , if there is a convex curve $C \subset \mathbb{R}P^{n-1}$ and three points $(t_1, t_2, t_3) \in \lambda_C$ such that $\mathbf{f}_i = \gamma_C(t_i)$. See *loc. cit.* for more details as well as a generalization to the complete flag variety of any split simple algebraic group over \mathbb{R} .

The concept of lexicographic order introduced above can be defined in the more general context of partial orders.

Definition 2.2.8. Let $f : I \rightarrow J$ be a map of sets. Suppose that J is equipped with a partial cyclic order λ_J , and each fiber $f^{-1}(j)$ is equipped with a partial linear order

\leq_j . In this setting we construct a partial cyclic order $\lambda = \text{Lex}(f, (\leq_j))$ on I , called the *lexicographic cyclic order* as follows. We say that $(a, b, c) \in \lambda_I$ if either

- (i) All three elements $f(a), f(b), f(c)$ are distinct and $(f(a), f(b), f(c)) \in \lambda_J$, or:
- (ii) $f(a) = f(b) \neq f(c)$ and $a \leq_{f(a)} b$, or:
- (ii') $f(a) = f(c) \neq f(b)$ and $c \leq_{f(c)} a$, or:
- (ii'') $f(a) \neq f(b) = f(c)$ and $b \leq_{f(b)} c$, or:
- (iii) $f(a) = f(b) = f(c)$ and either
 - (a) $a \leq_j b \leq_j c$, or
 - (b) $b \leq_j c \leq_j a$, or
 - (c) $c \leq_j a \leq_j b$, $j = f(a) = f(b) = f(c)$.

If λ_J is a total cyclic order, and each \leq_i is a total linear order, then $\text{Lex}(f, (\leq_j))$ is a total cyclic order.

Definition 2.2.9. Let (I, λ_I) and (J, λ_J) be partially cyclically ordered sets.

- (a) Assume that λ_I is a total cyclic order. Then we define a *morphism* $F : I \rightarrow J$ to be a datum consisting of a map $f : I \rightarrow J$ and a total linear order \leq_j on each $f^{-1}(j)$ such that $\text{Lex}(f, (\leq_j)) = \lambda_I$.
- (b) In general, a morphism $F : I \rightarrow J$ is a compatible system of morphisms $I' \rightarrow J$ given for all totally cyclically ordered subsets $I' \subset I$ (with respect to the induced cyclic order).

Note that the map of sets f underlying a morphism $F : I \rightarrow J$, is always “cyclically monotone”: $(a, b, c) \in \lambda_I$ implies $(f(a), f(b), f(c)) \in \lambda_J$. If f is injective, then F is determined by f . We call such morphisms *injective*. In general, however, a morphism contains more data than just this map f . The collection of partially cyclically ordered sets with morphisms as defined above forms a category $\mathbf{\Lambda}_{\text{par}}$ which contains the categories $\mathbf{\Lambda}$ and $\mathbf{\Lambda}$ as full subcategories.

2.2.C. Cyclic duality

Given a cyclically ordered set J of cardinality $n \geq 1$, the set $J^\vee := \text{Hom}_{\mathbf{\Lambda}}(J, \langle 0 \rangle)$ is, by definition, in canonical bijection with the set of *linear refinements* of the cyclic order on J , i.e., linear orders whose cyclic closure recovers the cyclic order on J . Further, the action of \mathbb{Z}/n on J induces a \mathbb{Z}/n -action on J^\vee which is simply transitive.

Example 2.2.10. Let $J \subset S^1$ be a subset of the standard circle. Providing J with the counter-clockwise cyclic order induced from S^1 , we may identify the set $\text{Hom}_\Lambda(J, \langle 0 \rangle)$ with the set of homotopy classes of monotone maps $f : S^1 \rightarrow S^1$ of degree 1 taking the subset J to $1 \in S^1$. Let us refer to the arcs on S^1 which connect an element of J to its successor as *interstices of J* . The map f is uniquely determined by specifying which interstice of J maps to a nontrivial loop in S^1 . Therefore, J^\vee is canonically identified with the set of interstices of $J \subset S^1$. For an abstract cyclically ordered set, the set J^\vee should be regarded as an intrinsic construction of the set of interstices with its cyclic order, without any reference to an embedding into the circle. The role of J^\vee as a dual object was emphasized by Drinfeld [19] whose result can be formulated as follows.

Proposition 2.2.11. *The cyclic interstice construction $J \mapsto J^\vee$ extends to a functor*

$$\Lambda^{\text{op}} \longrightarrow \Lambda$$

which is an equivalence of categories.

Proof. Although this is a particular case of our Proposition 2.1.22, let us provide a self-contained proof for convenience of the reader.

Given a morphism of cyclically ordered sets $f : J \rightarrow K$, we obtain a map of sets of linear refinements $f^\vee : K^\vee \rightarrow J^\vee$. To lift this map to a morphism of cyclically ordered sets, we have to additionally provide a linear order on each fiber of f^\vee . Consider linear orders $\lambda_1, \lambda_2 \in K^\vee$ which map to the same linear order $\lambda \in J^\vee$. Let j_{\max} be the maximum of (J, λ) . We say $\lambda_1 \leq \lambda_2$ if

$$|\{k \in K \mid f(j_{\max}) \leq_{\lambda_1} k\}| \leq |\{k \in K \mid f(j_{\max}) \leq_{\lambda_2} k\}|.$$

This construction provides linear orders on the fibers of f^\vee which are compatible with the cyclic order on K^\vee . The equivalence of categories can be verified by noting that the double dual $(J^\vee)^\vee$ can be canonically and functorially identified with J . \square

2.3 Dihedral orders

Let $\Delta\mathfrak{G} = \Xi$ be the dihedral category. A Ξ -structure on a set J yields a concept of *dihedral order*.

2.3.A. Finite total dihedral orders

Given a finite set J with cyclic order λ , we can define the *opposite* cyclic order λ^{op} on J by declaring the successor $j+1$ of an element j to be the predecessor $j-1$ of j with respect to λ . The association $(J, \lambda) \mapsto (J, \lambda^{\text{op}})$ extends to a self-equivalence of the category Λ where the opposite f^{op} of a morphism $f : J \rightarrow K$ is defined to be the same map on underlying sets equipped with the opposite linear order on each fiber.

Definition 2.3.1. A *dihedral order* on a finite set J of cardinality ≥ 3 is defined to be the choice of a set $o(J) = \{\lambda_0, \lambda_1\}$ of opposite cyclic orders on J . A dihedral order on a set J of cardinality ≤ 2 is defined to be the choice of an arbitrary two-element set $o(J)$. Given a dihedrally ordered set $(J, o(J))$, we refer to the two-element set $o(J)$ as the *orientation torsor* of J .

A morphism $J \rightarrow K$ of dihedrally ordered sets is defined to be a bijection $f : o(J) \rightarrow o(K)$ and an unordered pair $\{f_0 : (J, \lambda_0) \rightarrow (K, f(\lambda_0)), f_1 : (J, \lambda_1) \rightarrow (K, f(\lambda_1))\}$ of opposite morphisms of cyclic ordinals. Here, we agree by convention that for a dihedrally ordered set J of cardinality ≤ 2 , both elements of the orientation torsor $o(J)$ induce the unique cyclic order on J . With this notion of morphism, the set of finite nonempty dihedrally ordered sets organizes into a category Ξ which we call the *large dihedral category*.

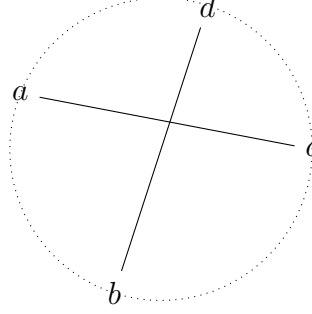
Remarks 2.3.2. (a) The reason for the introduction of a “virtual” orientation torsor for sets of cardinality ≤ 2 should now be clear. For example, with this convention, the automorphism groups of the standard dihedrally ordered sets $\{0\}$ and $\{0, 1\}$ are $D_1 = \mathbb{Z}/2$ and $D_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$, respectively. The nontrivial automorphism given by interchanging the elements of the virtual orientation torsor acts trivially on the underlying sets.

(b) The automorphism of $\mathbf{\Lambda}$ given by passing to the opposite cyclic order can be regarded as an action of the (discrete 2-) group $\mathbb{Z}/2$ on $\mathbf{\Lambda}$. Our approach amounts to defining Ξ as the quotient of $\mathbf{\Lambda}$ by this action.

Example 2.3.3. (1) Every set of cardinality $1 \leq n \leq 2$ admits a dihedral order which is unique up to nonunique isomorphism. This corresponds to the fact that we have surjections $D_n \rightarrow S_n$ with nontrivial kernel. Every set of cardinality 3 admits a unique dihedral order corresponding to the fact that we have an isomorphism $D_3 \rightarrow S_3$. A set of cardinality $n \geq 3$ has $|S_n|/|D_n|$ different dihedral orders.

(2) For a set J of cardinality ≥ 3 , a dihedral order on J can be identified with an equivalence class of embeddings of J into the standard unit circle $S^1 \subset \mathbb{C}$. Here two embeddings are equivalent if they differ by a (not necessarily orientation preserving) homeomorphism of S^1 . A set J of cardinality 4 has $3 = |S_4|/|D_4|$ dihedral orders. These orders are in natural bijection with unordered partitions of J into two 2-element sets: $J = \{a, c\} \sqcup \{b, d\}$. This can be immediately seen by identifying a dihedral order with an equivalence class of embeddings into S^1 : the above partition

encodes that the diagonals $[a, c]$ and $[b, d]$ intersect inside the unit disk.



Similarly, a dihedral order on a finite set J of cardinality ≥ 4 is uniquely determined by the knowledge of which pairs of diagonals intersect in the unit disk. In particular, this means that a dihedral order on J is uniquely recovered from the collection of the induced dihedral orders on all 4-element subsets of J .

- (3) Every finite cyclically ordered set (J, λ) admits a canonical dihedral order by setting $o(J) = \{\lambda, \lambda^{\text{op}}\}$ if $|J| \geq 3$ and $o(J) = \mathbb{Z}/2$ otherwise. We call this dihedral order the *dihedral closure* of λ . There is an obvious way to extend this construction to a faithful functor

$$\Lambda \longrightarrow \Xi$$

We denote the dihedral closure of the standard cyclically ordered set $\langle n \rangle$ by $\overline{\langle n \rangle}$. The small dihedral category Ξ is then obtained as the full subcategory of Ξ spanned by the standard dihedrally ordered sets $\{\overline{\langle n \rangle}\}$.

Reformulation 2.3.4. A more algebro-geometric version of Example 2.3.3 (2) can be obtained as follows. Let J be a finite set with ≥ 3 elements, and $\overline{M}_{0,J}$ be the moduli space of stable J -pointed curves of genus 0. This is a smooth projective variety over \mathbb{Q} of dimension $|J| - 3$ whose open part is the quotient

$$M_{0,J} = ((\mathbb{P}^1)^J \setminus (\text{diagonals})) / PGL_2,$$

see, e.g., [38] for more details. Now note that connected components of the real locus $M_{0,J}(\mathbb{R})$ are in bijection with dihedral orders on J . Indeed, $\mathbb{R}P^1 = \mathbb{P}^1(\mathbb{R})$ is topologically a circle, and action of $PGL_2(\mathbb{R})$ on it contains transformations both preserving and reversing the orientation of this circle, so a comparison as in Example 2.3.3 ((2)) is immediate. It was proved in [39] that the closure in $\overline{M}_{0,J}(\mathbb{R})$ of each component of $M_{0,J}(\mathbb{R})$, is identified with the Stasheff polytope $K_{|J|-1}$. Thus $\overline{M}_{0,J}(\mathbb{R})$ can be obtained by gluing as many Stasheff polytopes as there are dihedral orders on J .

2.3.B. General dihedral orders

According to Example 2.3.3 ((2)), a dihedral order on a finite set J can be understood as a 4-ary relation on J in analogy to Definition 2.2.5. In this paper we do not attempt a full

investigation of this approach which should naturally lead to a good concept of partial dihedral order on a possibly infinite set. Let us just indicate the most important property of the resulting 4-ary relation in the case of a finite dihedrally ordered set J . For any $n \geq 1$ let

$$J_{\#}^n = J^n \setminus \bigcup_{i \neq j} \{x_i = x_j\}$$

be the set of n -tuples of distinct elements of J , and let $\gamma \subset J_{\#}^4$ consist of 4-tuples (a, b, c, d) such that dihedral order on $\{a, b, c, d\}$ induced from the embedding into J coincides with the dihedral closure of the linear order (a, b, c, d) . For any $\mathbf{x} = (x_0, \dots, x_4) \in J_{\#}^5$ let

$$\partial_i \mathbf{x} = (x_0, \dots, \widehat{x_i}, \dots, x_4), \quad i = 0, \dots, 4,$$

be the tuple obtained by removing the i th element.

Proposition 2.3.5 (The pentagon property for dihedral orders). *Let J be a finite dihedrally ordered set, $|J| \geq 5$, and $\gamma \subset J_{\#}^4$ defined as above. For any $\mathbf{x} = (x_0, \dots, x_4) \in J_{\#}^5$ the following are equivalent:*

- (i) *All $\partial_i \mathbf{x}$, $i = 0, \dots, 4$, belong to γ . In other words, the total order on $\{x_0, \dots, x_4\}$ is compatible with the induced cyclic order.*
- (ii) *$\partial_1 \mathbf{x}$ and $\partial_3 \mathbf{x}$ belong to γ .*
- (iii) *$\partial_0 \mathbf{x}$, $\partial_2 \mathbf{x}$ and $\partial_4 \mathbf{x}$ belong to γ .*

Proof. Clearly, (i) implies (ii) and (iii). Let us prove that (ii) \Rightarrow (i). Let $J \hookrightarrow C$ be an embedding into a circle defining the dihedral order. Since $\partial_1 \mathbf{x} \in \gamma$, we have that x_0, x_2, x_3, x_4 are in the right dihedral order. In other words, their embedding into a circle C is such that they subdivide C into the following (unoriented) arcs: $A(x_0, x_2)$, $A(x_2, x_3)$, $A(x_3, x_4)$ and $A(x_4, x_0)$. Here $A(x_p, x_q)$ denotes the arc joining x_p, x_q which does not contain any other $x_j, j \in \{0, 2, 3, 4\}$. With respect to this embedding into C , the point x_1 must lie on one of these arcs, and (i) holds precisely when $x_1 \in A(x_0, x_2)$. Now, the assumption that $\partial_3 \mathbf{x} \in \gamma$ means that the diagonal $[x_0, x_2]$ crosses the diagonal $[x_1, x_4]$ which means that $x_1 \in A(x_0, x_2)$.

The proof that (iii) \Rightarrow (i) is similar. In fact, the conditions $\partial_1 \mathbf{x} \in \gamma$ and $\partial_3 \mathbf{x} \in \gamma$ already imply (i), the situation differing from (ii) by a cyclic rotation. \square

2.3.C. Dihedral duality.

Given any finite nonempty dihedrally ordered set J , consider the set $\widetilde{\mathcal{J}} := \text{Hom}_{\Xi}(J, \overline{\langle 0 \rangle})$. We can identify the set $\widetilde{\mathcal{J}}$ with the set of pairs (λ, o) where $o \in O(J)$ is an orientation and λ is a linear refinement of the cyclic order on J corresponding to o . In particular, $\widetilde{\mathcal{J}}$ admits a natural involution given by $(\lambda, o) \mapsto (\lambda^{\text{op}}, o^{\text{op}})$. We define J^{\vee} to be the orbit set under this involution. We can interpret $O(J)$ as an orientation torsor for J^{\vee} which determines opposite cyclic orders on J^{\vee} . Hence J^{\vee} admits a canonical dihedral order.

Example 2.3.6. Let $J \subset S^1$ be a subset of the standard circle considered without chosen orientation. Then J inherits a natural dihedral order and the morphism set $\text{Hom}_{\Xi}(J, \overline{\langle 0 \rangle})$ can be identified with homotopy classes of monotone homotopy equivalences $f : S^1 \rightarrow S^1$ which map the subset J to the point $1 \in S^1$ where we orient the target circle counter-clockwise. This map f is uniquely determined by the choice of an orientation and the choice of an interstice which maps to a nontrivial loop in S^1 . Therefore, $\text{Hom}_{\Xi}(J, \overline{\langle 0 \rangle})$ is in natural bijection with the set of *oriented interstices* of $J \subset S^1$. The quotient set $\text{Hom}_{\Xi}(J, \overline{\langle 0 \rangle})/(\mathbb{Z}/2)$ is then identified with the set of *unoriented interstices*. The two possible choice of coherent orientations of all interstices correspond to the elements of the orientation torsor.

Proposition 2.3.7. *The dihedral interstice construction $J \mapsto J^\vee$ extends to a functor*

$$\Xi^{\text{op}} \longrightarrow \Xi$$

which is an equivalence of categories. □

2.4 Paracyclic orders

Let $\Delta\mathfrak{G} = \Lambda_\infty$ be the paracyclic category. A Λ_∞ -structure on a set J defines the concept of a *paracyclic order*. As before, we study it at several levels of generality.

2.4.A. Finite total paracyclic orders

Definition 2.4.1. A *paracyclic order* on a finite nonempty set J consists of

- (1) a \mathbb{Z} -torsor \tilde{J} ,
- (2) a surjection $\pi : \tilde{J} \rightarrow J$ which exhibits J as the orbit set of the bijection

$$T : \tilde{J} \rightarrow \tilde{J}, \quad \tilde{j} \mapsto \tilde{j} + |J|.$$

Example 2.4.2. Let C be an oriented circle. A choice of a base point $c \in C$ defines a canonical paracyclic order on any finite nonempty $J \subset C$. More precisely, the choice of c produces a canonical universal covering $\pi : \tilde{C}_c \rightarrow C$, with \tilde{C}_c being homeomorphic to \mathbb{R} and oriented, and so equipped with a canonical total order. The discrete subset $\tilde{j} = \pi^{-1}(J)$ has a natural successor operation induced by the order on \tilde{C}_c , which makes it into a \mathbb{Z} -torsor. The generator of the group \mathbb{Z} of deck transformations on \tilde{C}_c corresponds to the shift by $|J|$ in this torsor structure.

A morphism $(J, \tilde{J}) \rightarrow (K, \tilde{K})$ of paracyclically ordered sets is given by a commutative diagram

$$\begin{array}{ccc} \tilde{J} & \xrightarrow{\tilde{f}} & \tilde{K} \\ \pi_J \downarrow & & \downarrow \pi_K \\ J & \xrightarrow{f} & K \end{array}$$

such that \tilde{f} is monotone with respect to the linear orders on \tilde{J} and \tilde{K} induced from the \mathbb{Z} -torsor structures. The resulting category of finite nonempty paracyclically ordered sets is called the *large paracyclic category* denoted by $\mathbf{\Lambda}_\infty$.

2.4.B. General paracyclic orders

Following [22, §3.1], call a \mathbb{Z} -ordered set as pair (S, F) , consisting of a partially ordered set (S, \leq) and an morphism of partially ordered set $F : S \rightarrow S$.

Definition 2.4.3. A *partial paracyclic order* on a set J is a datum of:

- (1) a \mathbb{Z} -ordered set (\tilde{J}, F) such that F is a fixed point-free bijection. We denote by $F^\mathbb{Z}$ the group of automorphisms of S generated by F .
- (2) A bijection $\tilde{J}/F^\mathbb{Z} \rightarrow J$.

Example 2.4.4. (Cf. [22], Ex. 3.1.5) Let \mathbf{k} be a discrete valued field with the ring of integers \mathfrak{o} and uniformizing element ϖ . Then the set $B = GL(n, \mathbf{k})/GL(n, \mathfrak{o})$ is identified with the set of \mathfrak{o} -lattices $L \subset \mathbf{k}^n$, so it is partially ordered by inclusion of lattices. The automorphism F given by multiplication of (matrices or lattices) by the scalar matrix ϖ , makes B into a \mathbb{Z} -ordered set. The quotient

$$B/\varpi^\mathbb{Z} = PGL(n, \mathbf{k})/PGL(n, \mathfrak{o})$$

has therefore a canonical (partial) paracyclic order.

2.4.C. Paracyclic duality.

We denote by pt the set $\{0\}$ equipped with the paracyclic order given by the trivial \mathbb{Z} -torsor. Given a finite nonempty paracyclically ordered set (J, \tilde{J}) , consider the set $\tilde{J}^\vee := \text{Hom}_{\mathbf{\Lambda}_\infty}(J, \text{pt})$ and define J^\vee to be the orbit set under the translation action of \mathbb{Z} on pt . We can equip \tilde{J}^\vee with the \mathbb{Z} -action given by pulling back the translation action on J . It is straightforward to verify that \tilde{J}^\vee is a \mathbb{Z} -torsor and the projection map $\tilde{J}^\vee \rightarrow J^\vee$ defines a paracyclic order on J^\vee .

Example 2.4.5. Let $J \subset S^1$ and consider the paracyclic order $\pi^{-1}(J) \rightarrow J$ on J induced from the universal cover map $\pi : \mathbb{R} \rightarrow S^1$ (corresponding to the base point $1 \in S^1$). Then we can identify $\text{Hom}_{\mathbf{\Lambda}_\infty}(J, \text{pt})$ with homotopy classes of monotone maps $\mathbb{R} \rightarrow \mathbb{R}$ which

- (1) map the subset $\tilde{J} \subset \mathbb{R}$ to the subset $\mathbb{Z} \subset \mathbb{R}$,
- (2) are equivariant with respect to translation by $|J|$ on \tilde{J} and translation by 1 on \mathbb{Z} .

Any such homotopy class is uniquely determined by specifying the interstice $[\tilde{j}, \tilde{j} + 1]$ of $\tilde{J} \subset \mathbb{R}$ which maps onto the interstice $[0, 1]$ of $\mathbb{Z} \subset \mathbb{R}$. Thus the set \tilde{J}^\vee can be naturally identified with the set of interstices of $\tilde{J} \subset \mathbb{R}$. The quotient J^\vee can therefore be canonically identified with the set of interstices of $J \subset S^1$.

Proposition 2.4.6. *The interstice construction $J \mapsto J^\vee$ extends to a functor*

$$(\Lambda_\infty)^{\text{op}} \longrightarrow \Lambda_\infty$$

which is an equivalence of categories.

□

3 Structured surfaces

3.1 Structured C^∞ -surfaces

Let $p : G \rightarrow O(2)$ be a planar Lie group, and $\mathbb{p} : \mathbb{G} = G^\circ \rightarrow GL(2, \mathbb{R})$ the connective covering extending p . We denote $K = \text{Ker}(p) = \text{Ker}(\mathbb{p})$. As the kernel of a connective covering, the group K is abelian. We say that \mathbb{G} *preserves orientation* if $\mathbb{p}(\mathbb{G}) \subset GL^+(2, \mathbb{R})$. Depending on whether \mathbb{G} preserves orientation or not, we have one of the following two short exact sequences of groups with abelian kernel:

$$(3.1.1) \quad \begin{aligned} 0 \rightarrow K \longrightarrow \mathbb{G} \longrightarrow GL^+(2, \mathbb{R}) \rightarrow 1, \\ 0 \rightarrow K \longrightarrow \mathbb{G} \longrightarrow GL(2, \mathbb{R}) \rightarrow 1. \end{aligned}$$

In the first case, K lies in the center of \mathbb{G} . In the second case, the action of \mathbb{G} on K by conjugation factors through an action of $\pi_0(GL(2, \mathbb{R})) = \{1, \tau\}$ given by an involution $\tau : K \rightarrow K$.

Proposition 3.1.2. (a) Let $\Delta\mathfrak{G}$ be the crossed simplicial group corresponding to \mathbb{G} . Then $K = \text{Ker}\{\rho : \mathfrak{G}_0 \rightarrow \mathbb{Z}/2\}$.

(b) The kernels corresponding to the various planar groups are given by:

G	$\text{Spin}_N(2)$	$\widetilde{SO(2)}$	$\text{Pin}_N^+(2)$	$\widetilde{O(2)}$	$\text{Pin}_{2M}^-(2)$
K	\mathbb{Z}/N	\mathbb{Z}	\mathbb{Z}/N	\mathbb{Z}	$\mathbb{Z}/2M$

If G does not preserve orientation then τ acts on K by $\tau(k) = -k$.

Proof. Follows from Proposition 1.6.1 (a) and (b) respectively. \square

By a *surface* we will mean a connected C^∞ -surface S possibly with boundary ∂S . Given a surface S , let $\text{Fr}_S \rightarrow S$ be the principal $GL(2, \mathbb{R})$ -bundle of frames in the tangent bundle T_S . The sheaf of connected components $\pi_0(\text{Fr}_S)$ is a $\mathbb{Z}/2$ -torsor over S which we call the *orientation torsor* of S and denote by or_S . Thus the total space of or_S is the orientation cover of S , denoted \tilde{S} .

Suppose \mathbb{G} does not preserve orientation. In this case, the $GL(2, \mathbb{R})$ -action on K from (3.1.1) together with the $GL(2, \mathbb{R})$ -torsor $\text{Fr}_S \rightarrow S$, gives rise to a local system of abelian groups $\underline{K}^{\text{or}}$ on S , locally isomorphic to \underline{K} . If S is orientable, then $\underline{K}^{\text{or}} \cong \underline{K}$ is a constant sheaf.

Definition 3.1.3. A \mathbb{G} -structured surface is a datum (S, F) , where S is a surface, and $F \xrightarrow{\rho} \text{Fr}_S$ is a reduction of structure groups along \mathbb{p} .

Thus $F \rightarrow S$ is a principal \mathbb{G} -bundle on S , and ρ is \mathbb{p} -equivariant.

Examples 3.1.4. We give examples using the notation of Table 3.

- (a) For $G = O(2)$, we have $\mathbb{G} = GL(2, \mathbb{R})$, and a \mathbb{G} -structured surface is just a surface with no additional structure.
- (b) For $G = SO(2)$, we have $\mathbb{G} = GL^+(2, \mathbb{R})$, and a \mathbb{G} -structure on S is the same as an orientation of S .
- (c) For $G = \text{Spin}_N(2)$, the group \mathbb{G} is the N -fold connected covering of $GL^+(2, \mathbb{R})$, and a \mathbb{G} -structure is known as an N -*spin structure*. The case $N = 2$, being an instance of the spinor construction existing in any number of dimensions, has been studied more systematically, cf. [37].
- (d) For the universal covering $G = \widetilde{SO(2)}$ of $SO(2)$, the group $\mathbb{G} = \widetilde{GL^+(2, \mathbb{R})}$ is the universal covering of $GL(2, \mathbb{R})$. A $\widetilde{GL^+(2, \mathbb{R})}$ -structure on a surface will be called a *framing*. Note that, in our definition, a framing on S is a discrete datum: not an actual choice of a trivialization of T_S but what amounts to an isotopy class of such trivializations. In fact, the same is true in all the other cases, and this is the reason for working with \mathbb{G} instead of G . For instance, an $O(2)$ -structure on S is a Riemannian metric, while a $GL(2, \mathbb{R})$ -structure is no extra structure at all.

A *structured diffeomorphism* of \mathbb{G} -structured surfaces $(S, F) \rightarrow (S', F')$ is a pair $(\phi, \tilde{\phi})$ where $\phi : S \rightarrow S'$ is a diffeomorphism, and $\tilde{\phi} : F \rightarrow F'$ is a \mathbb{G} -equivariant diffeomorphism lifting $d\phi : \text{Fr}_S \rightarrow \text{Fr}_{S'}$. A structured diffeomorphism $(S, F) \rightarrow (S, F')$ with $\phi = \text{Id}$ will be called an *isomorphism of \mathbb{G} -structures* on S . We denote by $\mathbb{G}\text{Str}(S)$ the groupoid formed by \mathbb{G} -structures on S and their isomorphisms.

Suppose \mathbb{G} preserves orientation. Then any \mathbb{G} -structure on S gives rise to an orientation. Given an oriented surface S , we denote by $\mathbb{G}\text{Str}^+(S)$ the full subcategory in $\mathbb{G}\text{Str}(S)$ formed by those \mathbb{G} -structures which induce the given orientation.

The categories $\mathbb{G}\text{Str}(S), \mathbb{G}\text{Str}^+(S)$ can be understood in terms of standard constructions of non-abelian cohomology [31]. For a Lie group L , let \underline{L} be the sheaf of groups on S formed by local C^∞ -maps into L . Thus, the pointed set $H^1(S, \underline{L})$ is the set of isomorphism classes of principal L -bundles on S .

Proposition 3.1.5. (a) *Suppose \mathbb{G} preserves orientation, and let S be an oriented surface. Then:*

- (a1) *For any object $F \in \mathbb{G}\text{Str}^+(S)$, the group $\text{Aut}(F)$ is identified with K .*
- (a2) *The set of isomorphism classes of objects of $\mathbb{G}\text{Str}^+(S)$ is either empty or a torsor over $H^1(S, K)$.*

(b) *Suppose \mathbb{G} does not preserve orientation, and let S be any surface. Then:*

- (b1) *For any object $F \in \mathbb{G}\text{Str}(S)$, the group $\text{Aut}(F)$ is identified with $H^0(S, \underline{K}^{\text{or}})$.*
- (b2) *The set of isomorphism classes of objects of $\mathbb{G}\text{Str}(S)$ is either empty, or is a torsor over $H^1(S, \underline{K}^{\text{or}})$.*

(b3) The group $H^0(S, \underline{K}^{\text{or}})$ is identified with K , if S is orientable, and with the 2-torsion subgroup $K_2 \subset K$, if S is non-orientable.

Proof. Let X be a topological space and

$$0 \rightarrow A \longrightarrow \Gamma' \xrightarrow{\pi} \Gamma \rightarrow 1$$

be an extension of sheaves of groups on X with abelian kernel A . Let P be a sheaf of Γ -torsors on X , with associated class $[P] \in H^1(X, \Gamma)$. In this situation, the Γ -action on A by conjugation gives rise to the twisted form A^P , a sheaf of abelian groups on X locally isomorphic to A . Denote by $\text{Lift}(P)$ the category formed by sheaves P' of Γ' -torsors on X together with a π -equivariant map $\pi_{P'} : P' \rightarrow P$. Morphisms in $\text{Lift}(P)$ are isomorphisms of torsors commuting with the projection to P . In the described situation, we have:

- (i) Each group $\text{Aut}_{\text{Lift}(P)}(P')$ is identified with $H^0(X, A^P)$.
- (ii) The set of isomorphism classes of objects of $\text{Lift}(P)$ is either empty or is a torsor over $H^1(X, A^P)$.
- (iii) If A is central, then there is a natural coboundary map of pointed sets $\delta : H^1(X, \Gamma) \rightarrow H^2(X, A)$. In this case $\text{Lift}(P) \neq \emptyset$ if and only if $\delta([P]) = 0$.

Now, parts (a1-2) and (b1-2) follow by considering the extensions of sheaves on S obtained from (3.1.1) by passing to sheaves of C^∞ -sections. Part (b3) follows from Proposition 3.1.2(b). \square

Example 3.1.6. Let S be an oriented surface and $G = \text{Spin}_N(2)$. If S is noncompact or has nonempty boundary then we have $H^2(S, K) = 0$ so that $\mathbb{G}\text{Str}^+(S)$ is nonempty. If S is compact without boundary, then $\mathbb{G}\text{Str}^+(S)$ is nonempty for $N = 2$. This follows by considering the inclusion of short exact sequences of sheaves of groups

$$(3.1.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}/2} & \longrightarrow & \underline{S^1} & \xrightarrow{z \mapsto z^2} & \underline{S^1} \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{\mathbb{Z}/2} & \longrightarrow & \underline{GL^+(2, \mathbb{R})}_2 & \longrightarrow & \underline{GL^+(2, \mathbb{R})} \longrightarrow 1 \end{array}$$

where $S^1 = SO(2)$. Choosing a Riemannian metric on S , we reduce the structure group to S^1 and so realize $[\text{Fr}_S]$ as coming from an element of $H^1(S, \underline{S^1})$. The coboundary map $\delta_{S^1} : H^1(S, \underline{S^1}) \rightarrow H^2(S, \underline{\mathbb{Z}/2}) \cong \underline{\mathbb{Z}/2}$ is given by the degree modulo 2. Since the tangent bundle of a surface of genus g has degree $2g - 2$, we have $\delta_{S^1}([\text{Fr}_S]) = 0$.

Definition 3.1.8. A \mathbb{G} -structured marked surface is a datum consisting of:

- (1) A compact surface S (possibly with boundary). We denote $S^\circ = S - \partial S$ the interior of S .

- (2) A finite subset $M \subset S$.
- (3) A \mathbb{G} -structure F on $S - (M \cap S^\circ)$.

We will usually denote \mathbb{G} -structured marked surfaces by (S, M) , keeping the \mathbb{G} -structure implicit.

Definition 3.1.9. Let (S, M) and (S', M') be two \mathbb{G} -structured marked surfaces, with \mathbb{G} -structured torsors F and F' respectively. A *structured diffeomorphism* $(S, M) \rightarrow (S', M')$ is a datum consisting of:

- (1) A diffeomorphism $\phi : S \rightarrow S'$ taking M to M' .
- (2) A structured diffeomorphism $\tilde{\phi} : S - (M \cap S^\circ) \rightarrow S' - (M' \cap S'^\circ)$ lifting the restriction of ϕ .

Note that ϕ in (1) is uniquely determined by $\tilde{\phi}$ in (2).

For a \mathbb{G} -structured marked surface (S, M) , we denote by $\text{Diff}^\mathbb{G}(S, M)$ the group of \mathbb{G} -structured self-diffeomorphisms of (S, M) , and by $\text{Diff}_{\text{pure}}^\mathbb{G}(S, M)$ the subgroup of structured self-diffeomorphisms preserving M pointwise. We consider these groups as subgroups in the group of diffeomorphisms of the structure torsor F and equip them with the subspace topology.

For $\mathbb{G} = GL(2, \mathbb{R})$, we get the ordinary (pure) diffeomorphism group of the marked surface (S, M) , which we denote $\text{Diff}(S, M)$ and $\text{Diff}_{\text{pure}}(S, M)$. Forgetting the \mathbb{G} -structured data gives morphisms of topological groups

$$\begin{aligned} \pi : \text{Diff}^\mathbb{G}(S, M) &\longrightarrow \text{Diff}(S, M), \\ \pi_{\text{pure}} : \text{Diff}_{\text{pure}}^\mathbb{G}(S, M) &\longrightarrow \text{Diff}_{\text{pure}}(S, M). \end{aligned}$$

Proposition 3.1.10. *The group $\text{Ker}(\pi) = \text{Ker}(\pi_{\text{pure}})$ is a discrete abelian group, so that π and π_{pure} are unramified coverings. Further, the groups $\text{Im}(\pi)$ and $\text{Im}(\pi_{\text{pure}})$ are unions of connected components of $\text{Diff}(S, M)$.*

Proof. The group $\text{Ker}(\pi) = \text{Ker}(\pi_{\text{pure}})$ is identified with $\text{Aut}(F)$, the group of automorphisms of the given \mathbb{G} -structure F on $S - (M \cap S^\circ)$. This group was described in Proposition 3.1.5 (a1)(b1), in particular it is abelian. Further, $\phi \in \text{Diff}(S, M)$ lies in $\text{Im}(\pi)$ if and only if the \mathbb{G} -structure ϕ^*F on $S - (M \cap S^\circ)$ is isomorphic to F . So $\text{Im}(\pi)$ is the stabilizer of the action of $\text{Diff}(S, M)$ on the set of isomorphism classes of objects of $\mathbb{G}\text{Str}(S - (M \cap S^\circ))$. This set was described in Proposition 3.1.5 (a2)(b2), in particular, it is discrete. So $\text{Im}(\pi)$ is open and closed, and similarly for $\text{Im}(\pi_{\text{pure}})$. \square

Definition 3.1.11. Let (S, M) be a \mathbb{G} -structured marked surface. We define the \mathbb{G} -structured mapping class groups as

$$\text{Mod}^\mathbb{G}(S, M) = \pi_0 \text{Diff}^\mathbb{G}(S, M), \quad \text{Mod}_{\text{pure}}^\mathbb{G}(S, M) = \pi_0 \text{Diff}_{\text{pure}}^\mathbb{G}(S, M).$$

The following classical construction will be used frequently.

Definition 3.1.12. Let S be a compact C^∞ -surface with boundary ∂S , and $S^\circ = S - \partial S$. The *Schottky double* of S is the compact oriented C^∞ surface $S^\#$ without boundary obtained by compatifying $\widetilde{S^\circ}$, the orientation cover of S° , by adding one copy of ∂S . Thus we have a 2-sheeted covering $\pi : S^\# \rightarrow S$, ramified along ∂S , and the deck involution $\sigma : S^\# \rightarrow S^\#$ of this covering, reversing the orientation and identical on ∂S . If (S, M) is a marked surface, we equip $S^\#$ with the subset $M^\# = \pi^{-1}(M)$.

Note that $S^\#$ is not the same as \widetilde{S} , if $\partial S \neq \emptyset$.

Definition 3.1.13. [60] A surface S is called *classical*, if S is orientable and $\partial S = \emptyset$, and *non-classical* otherwise.

Thus $S^\#$ is connected, if and only if S is non-classical.

Definition 3.1.14. A marked surface (S, M) is called *stable*, if:

- (1) $M \neq \emptyset$ and moreover, M meets each component of ∂S .
- (2) $\chi(S^\# - M^\#) < 0$.

For orientable surfaces this is equivalent to Def. 3.11 of [23]. For a stable marked surface (S, M) the points of M lying in the interior of S , will be called *punctures*. By blowing up each puncture to a small circle, one can obtain an alternative point of view on stable marked surfaces, in which all marked points are supposed to lie on the boundary, but it is allowed to have boundary components without a marked point. See [23, Rem. 3.3.2].

Proposition 3.1.15. *If (S, M) is a stable \mathbb{G} -structured marked surface, then each component of $\text{Diff}^\mathbb{G}(S, M)$ is contractible.*

Proof. By Proposition 3.1.10, the statement reduces to the particular case of $\mathbb{G} = GL(2, \mathbb{R})$ where it translates to the contractibility of each component or, equivalently, of the identity component of $\text{Diff}(S, M)$. This statement is shown in [32] [for compact with boundary, but this seems to give the statement, since we can just interpret a boundary component as a marked point by gluing in a center-marked disk?] \square

3.2 Conformal structured surfaces

3.2.A. The setup

As in §3.1, we fix a planar Lie group G and the corresponding connective covering $\mathbb{G} \xrightarrow{\mathbb{P}} GL_2(\mathbb{R})$. Let

$$(3.2.1) \quad \text{Conf}(2) = (\mathbb{Z}/2) \ltimes \mathbb{C}^* \subset GL(2, \mathbb{R})$$

be the subgroup of conformal transformations. Since the embedding (3.2.1) is a homotopy equivalence, the categories of connective coverings of $GL(2, \mathbb{R})$ and $\text{Conf}(2)$ are equivalent by Proposition 1.5.3(c). We denote

$$p_{\text{conf}} : G_{\text{conf}} \longrightarrow \text{Conf}(2)$$

the connective covering obtained by restricting \mathfrak{p} .

Similarly to §3.1, by a G_{conf} -structured surface we mean a surface S together with a restriction of the structure group of Fr_S to G_{conf} . As before, such a structure is a datum of a principal G_{conf} -bundle F_{conf} together with a compatible P_{conf} -equivariant map $F_{\text{conf}} \rightarrow \text{Fr}_S$. The concepts of a structured diffeomorphism between two G_{conf} -structured surfaces and of an isomorphism between two G_{conf} -structures on a given surface S are defined entirely similar to §3.1. In particular, we have a groupoid $G_{\text{conf}} \text{Str}(S)$ of G_{conf} -structures on S and their isomorphisms. If \mathbb{G} preserves orientation and S is oriented, we also have the full subcategory $G_{\text{conf}} \text{Str}^+(S)$ of G_{conf} -structures compatible with the orientation.

Examples 3.2.2. (a) An $SO(2)_{\text{conf}}$ -structured surface is a *Riemann surface*, i.e., a 1-dimensional complex manifold X possibly with boundary. More precisely, an isomorphism class of objects of $SO(2)_{\text{conf}} \text{Str}(S)$ is the same as an almost complex structure $J : T_S \rightarrow T_S$, $J^2 = -1$, and the automorphism group of any object of $SO(2)_{\text{conf}} \text{Str}(S)$ is trivial. As well known, all almost complex structures in complex dimension 1 are integrable. For a Riemann surface $X = (S, J)$ we denote by $\bar{X} = (S, -J)$ the *conjugate* Riemann surface.

(b) A $\text{Spin}_N(2)_{\text{conf}}$ -structured surface is a triple (X, \mathcal{L}, ϕ) where X is a Riemann surface, \mathcal{L} is a line bundle on X , and $\phi : \mathcal{L}^{\otimes N} \rightarrow \omega_X$ is an isomorphism. Here ω_X is the canonical line bundle of X . For $N = 2$, such line bundles \mathcal{L} are traditionally called *theta-characteristics*, and a choice of \mathcal{L} is often referred to as a “spin structure” on X , cf. [4]. For “higher spin structures” on Riemann surfaces ($N > 2$), see [35].

(c) An $O(2)_{\text{conf}}$ -structured surface is a *Klein surface*, i.e., a surface together with an atlas whose transition maps are conformal but do not necessarily preserve orientation. Such maps are called *dianalytic* (i.e., analytic or anti-analytic). Thus a Klein surface does not have to be orientable. See [2, 10] for more background.

Proposition 3.2.3. (a) For any surface S we have a natural identification of groupoids

$$G_{\text{conf}} \text{Str}(S) \simeq \mathbb{G} \text{Str}(S) \times O(2)_{\text{conf}} \text{Str}(S).$$

(b) The automorphism group of any object of $O(2)_{\text{conf}} \text{Str}(S)$ is trivial.

Proof. (a) By definition, we have

$$G_{\text{conf}} = \mathbb{G} \times_{GL(2, \mathbb{R})} O(2)_{\text{conf}}$$

where $O(2)_{\text{conf}} = \text{Conf}(2)$. So given a \mathbb{G} -structure $F \rightarrow \text{Fr}_S$ and an $O(2)_{\text{conf}}$ -structure $F_{\text{conf}}^{O(2)} \rightarrow \text{Fr}_S$, we obtain a G_{conf} -structure by forming the fiber product

$$F_{\text{conf}} = F \times_{\text{Fr}_S} F_{\text{conf}}^{O(2)}$$

which is a principal G_{conf} -bundle. We leave the remaining details to the reader.

(b) Follows because $O(2)_{\text{conf}}$ is a subgroup of $GL(2, \mathbb{R})$. \square

3.2.B. Klein surfaces and algebraic curves over \mathbb{R}

Let X be a Klein surface, and $X^\circ = X - \partial X$. For each $x \in X^\circ$, there are exactly two complex structures on $T_x X$ compatible with the given conformal structure. The set of all these structures for all $x \in X^\circ$ is canonically identified with the orientation cover \widetilde{X}° . Thus \widetilde{X}° has a canonical complex structure, making it into a (possibly non-compact) Riemann surface. Further, the complex structure on \widetilde{X}° extends naturally to the Schottky double $X^\# = \widetilde{X}^\circ \sqcup \partial X$. The local structure of $X^\#$ near the boundary is clear from the following example.

Example 3.2.4. Let X be a Riemann surface. Then $X^\# = X \cup_{\partial X} \overline{X}$.

Thus $X^\#$, being a compact Riemann surface without boundary, can be considered as a smooth projective algebraic curve over \mathbb{C} . Further, the canonical involution $\sigma : X^\# \rightarrow X^\#$ is anti-holomorphic and makes $X^\#$ defined over \mathbb{R} , so that the real locus of $X^\#$ is $X^\#(\mathbb{R}) = \partial S$. The following is classical [2, 54, 10].

Proposition 3.2.5. *The Schottky double construction establishes an equivalence between the following categories:*

- (i) *Non-classical Klein surfaces X .*
- (ii) *Pairs (Y, σ) where Y is a compact Riemann surface without boundary and $\sigma : Y \rightarrow Y$ is an anti-holomorphic involution.*
- (iii) *Smooth projective algebraic curves over \mathbb{R} .* \square

Pairs (Y, σ) as in (ii) are known as *symmetric Riemann surfaces*. The Klein surface corresponding to a symmetric Riemann surface (Y, σ) is the orbit space Y/σ .

3.2.C. Teichmüller spaces of marked Klein surfaces

Let (S, M) be an oriented marked surface without boundary. We denote by $\text{Riem}^+(S)$ the space of all complex structures on S compatible with the orientation. Let $\text{Diff}(S, M)_e$ be the identity component of $\text{Diff}(S, M)$. Diffeomorphisms in this component preserve the orientation, so $\text{Diff}(S, M)_e$ acts on $\text{Riem}^+(S)$. The classical *Teichmüller space* of (S, M) is defined as the quotient

$$\text{Teich}^+(S, M) = \text{Diff}(S, M)_e \backslash \text{Riem}^+(S).$$

The following is well known.

Theorem 3.2.6. *Let g be the genus of S and suppose that (S, M) is stable, i.e., $2g - 2 + |M| > 0$. Then:*

- (a) *$\text{Teich}^+(S, M)$ has a natural structure of a complex manifold of dimension $3g - 3 + |M|$. As a C^∞ -manifold, it is diffeomorphic to Euclidean space.*
- (b) *Let $\sigma : S \rightarrow S$ be an orientation reversing involution preserving M as a set. Conjugation with σ defines an anti-holomorphic involution*

$$\sigma_{\text{Teich}} : \text{Teich}^+(S, M) \longrightarrow \text{Teich}^+(S, M). \quad \square$$

Let now (S, M) be an arbitrary stable marked surface. Denote by $\text{Klein}(S)$ the space of all possible conformal structures on S . The *Teichmüller space* of (S, M) is defined by

$$\text{Teich}(S, M) = \text{Diff}(S, M) \setminus \text{Klein}(S).$$

Theorem 3.2.7. (a) *If S is classical (i.e., orientable without boundary), then the space $\text{Teich}(S, M)$ is the disjoint union of two connected components, corresponding to the two orientations of S . Each component is canonically identified with the corresponding classical Teichmüller space $\text{Teich}^+(S, M)$.*

- (b1) *Suppose S is non-classical, let $\pi : S^\# \rightarrow S$ be its Schottky double with involution σ , and let $M^\# = \pi^{-1}(M)$. Let $g^\#$ be the genus of $S^\#$. Then*

$$\text{Teich}(S, M) = (\text{Teich}^+(S^\#, M^\#))^{\sigma_{\text{Teich}}}$$

is the fixed point locus of the anti-holomorphic involution σ_{Teich} . In particular, it is a real analytic manifold of dimension $3g^\# - 3 + |M^\#|$.

- (b2) *Suppose S is non-orientable, let $\varpi : \tilde{S} \rightarrow S$ be its orientation cover with the orientation reversing deck involution τ , and let $\tilde{M} = \varpi^{-1}(M)$. Then*

$$\text{Teich}(S, M) = (\text{Teich}^+(\tilde{S}, \tilde{M}))^{\tau_{\text{Teich}}}$$

is the fixed point locus of the involution τ_{Teich} .

- (c) *In the situation (b1), the manifold $\text{Teich}(S, M)$ is diffeomorphic to Euclidean space.*

Proof. Parts (a), (b1-2) are obvious. Part (c) was proved by Natanzon [54] for $\partial S = M = \emptyset$ and Seppälä [60] for $M = \emptyset$. The extension to the case $M \neq \emptyset$ can be done by the same methods. \square

3.2.D. Moduli spaces of marked G_{conf} -structured surfaces

Definition 3.2.8. A G_{conf} -structured marked surface is a datum consisting of:

- (1) A conformal surface S .
- (2) A finite subset $M \subset S$.
- (3) A G_{conf} -structure on $S - (M \cap S^\circ)$ compatible with the conformal structure restricted from S .

Given a C^∞ marked surface (S, M) , we have a topological groupoid $G_{\text{conf}} \text{Str}(S, M)$ formed by pairs consisting of a conformal structure on S and a compatible G_{conf} -structure on $S - (M \cap S^\circ)$. By Proposition 3.2.3 we have

$$(3.2.9) \quad G_{\text{conf}} \text{Str}(S, M) \simeq \mathbb{G} \text{Str}(S - (M \cap S^\circ)) \times O(2)_{\text{conf}} \text{Str}(S).$$

Let (S, M) be a stable marked surface. The *moduli spaces* of G_{conf} -structures on (S, M) are defined as the groupoid quotients

$$(3.2.10) \quad \begin{aligned} \mathcal{M}^{G_{\text{conf}}}(S, M) &= \text{Diff}(S, M) \backslash\!\!\backslash G_{\text{conf}} \text{Str}(S, M), \\ \mathcal{M}_{\text{pure}}^{G_{\text{conf}}}(S, M) &= \text{Diff}_{\text{pure}}(S, M) \backslash\!\!\backslash G_{\text{conf}} \text{Str}(S, M). \end{aligned}$$

Thus they are topological groupoids, and $\mathcal{M}^{G_{\text{conf}}}(S, M)$ is a further quotient of $\mathcal{M}_{\text{pure}}^{G_{\text{conf}}}(S, M)$ by a finite subgroup of permutations of M .

Remark 3.2.11. Equivalently, $\mathcal{M}^{G_{\text{conf}}}(S, M)$ can be seen as the groupoid formed by all G_{conf} -structured marked surfaces of topological type (S, M) and by their structured diffeomorphisms.

Example 3.2.12. Let M be a finite set and suppose that $2g - 2 + |M| > 0$. Let $\mathcal{M}(g, M)$ be the Deligne-Mumford moduli stack formed by smooth projective algebraic curves X of genus g together with an embedding $M \hookrightarrow X$. Let (S, M) be a classical marked surface of genus g . Any choice of an orientation of S identifies $\mathcal{M}_{\text{pure}}^{SO(2)_{\text{conf}}}(S, M)$ with the groupoid (orbifold) of \mathbb{C} -points $\mathcal{M}(g, M)(\mathbb{C})$. Change of orientation corresponds to the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathcal{M}(g, M)(\mathbb{C})$ which comes from the fact that $\mathcal{M}(g, M)$ is defined over \mathbb{R} (in fact, over \mathbb{Q}). The orbifold $\mathcal{M}^{SO(2)_{\text{conf}}}$ is therefore the quotient of $\mathcal{M}(g, M)(\mathbb{C})$ by the symmetric group of M . As is well known,

$$\mathcal{M}(g, M)(\mathbb{C}) = \text{Mod}_{\text{pure}}^+(S, M) \backslash\!\!\backslash \text{Teich}^+(S, M)$$

is the quotient of the topological cell $\text{Teich}^+(S, M)$ by the mapping class group of orientation preserving diffeomorphisms fixing all points of M . In what follows, we realize a general $\mathcal{M}^{G_{\text{conf}}}(S, M)$ in terms of quotients of cells by discrete groups.

Let (S, M) be stable, and define

$$(3.2.13) \quad \mathcal{M}^{\mathbb{G}}(S, M) = \text{Diff}(S, M) \amalg \mathbb{G} \text{Str}(S - (M \cap S^\circ)) \simeq \text{Mod}(S, M) \amalg \mathbb{G} \text{Str}(S - (M \cap S^\circ))$$

as the “topological” analog of the moduli space, in which we consider \mathbb{G} instead of G_{conf} as a structure group. Here the last equivalence is a homotopy equivalence of topological groupoids coming from contractibility of $\text{Diff}(S, M)_e$ (Proposition 3.1.15). So we can and will consider $\mathcal{M}^{\mathbb{G}}(S, M)$ as a set-theoretic, non-topological, groupoid given by the right-hand side of (3.2.13). Similarly, we define $\mathcal{M}_{\text{pure}}^{\mathbb{G}}(S, M)$, using $\text{Mod}_{\text{pure}}(S, M)$.

Proposition 3.2.14. *We have a homotopy equivalence of topological groupoids*

$$\mathcal{M}^{G_{\text{conf}}}(S, M) \simeq \coprod_{\{F\} \in \mathcal{M}^{\mathbb{G}}(S, M)} \text{Mod}^{\mathbb{G}}(S_F, M) \amalg \text{Teich}(S, M)$$

and a similar identification of $\mathcal{M}_{\text{pure}}^{G_{\text{conf}}}(S, M)$. Here $\{F\}$ runs over isomorphism classes of objects of $\mathcal{M}^{\mathbb{G}}(S, M)$, and S_F is the \mathbb{G} -structured surface corresponding to F .

Proof. Using (3.2.9), we have

$$\begin{aligned} \mathcal{M}^{G_{\text{conf}}}(S, M) &= \text{Diff}(S, M) \amalg (\mathbb{G} \text{Str}(S - (M \cap S^\circ)) \times O(2)_{\text{conf}} \text{Str}(S)) \\ &= \text{Diff}(S, M) \amalg \coprod_{\{F\} \in \mathbb{G} \text{Str}(S - (M \cap S^\circ))} O(2)_{\text{conf}} \text{Str}(S) \\ &= \coprod_{\{F\} \in \text{Diff}(S, M) \amalg \mathbb{G} \text{Str}(S - (M \cap S^\circ))} \left(\text{Stab}(F) \amalg O(2)_{\text{conf}} \text{Str}(S) \right). \end{aligned}$$

Note that at the bottom of last coproduct the quotient by $\text{Diff}(S, M)$ is $\mathcal{M}^{\mathbb{G}}(S, M)$. At the same time, $\text{Stab}(F)$ is the union of connected components of $\text{Diff}(S, M)$, in particular, $\text{Stab}(F)_e = \text{Diff}(S, M)_e$. Therefore the above coproduct can be identified with

$$\begin{aligned} \coprod_{\{F\} \in \mathcal{M}^{\mathbb{G}}(S, M)} \pi_0(\text{Stab}(F)) \amalg \left(\text{Stab}(F)_e \amalg O(2)_{\text{conf}} \text{Str}(S) \right) &= \\ &= \coprod_{\{F\} \in \mathcal{M}^{\mathbb{G}}(S, M)} \text{Mod}^{\mathbb{G}}(S_F, M) \amalg \text{Teich}(S, M) \end{aligned}$$

as claimed. In the last identification, we identified the topological groupoid $O(2)_{\text{conf}} \text{Str}(S)$ with the topological space $\text{Klein}(S)$ of its isomorphism classes. This is possible because all automorphism groups of objects in this groupoid are trivial by Proposition 3.2.3(b). \square

4 Structured graphs

4.1 Structured graphs and structured surfaces

Definition 4.1.1. A *graph* Γ is a pair of sets (H, V) equipped with

- an involution $\tau : H \rightarrow H$,
- a map $s : H \rightarrow V$.

The elements of H are called *halfedges* of Γ , the fixed points of τ are called *external halfedges*, and the nonfixed points of τ are called *internal halfedges*. The 2-element orbits of τ , which are hence comprised of a pair of internal halfedges, are called *edges*. The elements of V are called the *vertices* of Γ . Given a vertex $v \in V$, we define the set $H(v) = s^{-1}(v)$ of *halfedges incident to v* . The cardinality $|H(v)|$ is called the *valency* of v .

Definition 4.1.2. Let Γ be a graph. The *incidence category* $I(\Gamma)$ of Γ has, as objects, all the vertices and edges of Γ with a non-identity morphism $s(h) \rightarrow \{h, \tau(h)\}$ for every internal halfedge h . The *incidence diagram* of Γ is the functor

$$I_\Gamma : I(\Gamma) \longrightarrow \mathcal{S}et$$

given on objects by assigning to a vertex v the set $H(v)$ and to an edge e with half-edges h, h' , the 2-element set $\{h, h'\}$. To a morphism $s(h) \rightarrow \{h, h'\}$ we associate the map of sets $H(s(h)) \rightarrow \{h, h'\}$ given by mapping h to h' and collapsing the remaining halfedges in $H(s(h))$ (denoted by A in Fig. 1) to h .

We say a graph is *compact* if all its halfedges are internal. Unless stated otherwise, we will assume all graphs to be compact.

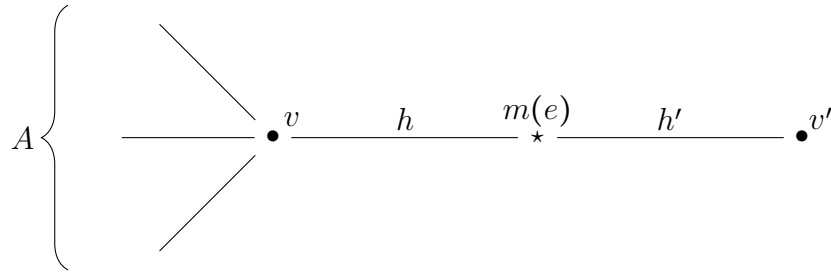


Figure 1: The incidence map: $m(e)$ is the midpoint of the edge $e = \{h, h'\}$.

Definition 4.1.3. Given a graph Γ , we define its *realization* $|\Gamma| = |N(I(\Gamma))|$ to be the geometric realization of its incidence category. We denote by $\partial|\Gamma| \subset |\Gamma|$ the set of 1-valent vertices and by $|\Gamma|^\circ$ the complement $|\Gamma| - \partial|\Gamma|$.

Remark 4.1.4. Note that the realization $|\Gamma|$ comes equipped with two kinds of 0-cells. On the one hand, we have a 0-cell for each vertex v of Γ , on the other hand, we have a 0-cell $m(e)$ for each edge e of Γ which can be regarded as a chosen midpoint of the geometric edge which connects the two vertices of Γ incident to e , see Figure 1.

Definition 4.1.5. Let Γ be a graph and let \mathcal{G} be a small category equipped with a functor $\mathcal{G} \rightarrow \mathcal{S}et$. A \mathcal{G} -structure on Γ is a lift

$$\begin{array}{ccc} & & \mathcal{G} \\ & \nearrow \tilde{I}_\Gamma & \downarrow \\ I(\Gamma) & \xrightarrow{I_\Gamma} & \mathcal{S}et \end{array}$$

of the incidence diagram of Γ to \mathcal{G} .

We now fix a planar Lie group $\rho : G \rightarrow O(2)$, the corresponding crossed simplicial group $\Delta\mathfrak{G}$, and the larger (but equivalent) category \mathcal{G} of $\Delta\mathfrak{G}$ -structured sets. Note that the interpretation of the objects of \mathcal{G} as sets with extra structure gives a forgetful functor $\mathcal{G} \rightarrow \mathcal{S}et$. In this context, we will slightly abuse notation, and refer to \mathcal{G} -structured graphs as $\Delta\mathfrak{G}$ -structured graphs.

Examples 4.1.6. (1) A graph with a Λ -structure is called a *Ribbon graph*. Explicitly, a Λ -structure on a graph Γ is the datum of a cyclic order on the $H(v)$ for each vertex $v \in V$. Each set $\{h, h'\}$ of half-edges of any edge e has a trivial cyclic order, since it has cardinality 2. Thus, our definition reduces to the usual one [57].

(2) A graph with a Ξ -structure is called a *Möbius graph*. Explicitly, a Ξ -structure on a graph Γ consists, first, of a dihedral order on each $H(v)$ and, second, of identification of the orientation torsors

$$O(H(v)) \xleftarrow{\sim} O(e) \xrightarrow{\sim} O(H(v'))$$

for each edge e with vertices v and v' (the case $v = v'$ is allowed). This structure is equivalent to the (somewhat more cumbersome) concept of a Möbius graph defined in terms of “ribbon graphs with $\mathbb{Z}/2$ -grading on edges” as in [10, 52, 53].

(3) A graph with a Λ_∞ -structure is called a *framed graph*.

(4) A graph with a Λ_N -structure is called an *N -spin graph*. Very recently, surfaces with 2-spin structure were studied, from a combinatorial point of view by Novak and Runkel [56]. They introduced a concept of a combinatorial spin structure on a triangulation T of a surface S . Such a structure equips the dual graph of T with a Λ_2 -structure in our sense.

For a 2-dimensional \mathbb{R} -vector space V , we denote by

$$C(V) = (V - \{0\})/\mathbb{R}_{>0}^*$$

the circle of directions of V . In particular, for any surface S and any $x \in S$ we have the circle $C(T_x S)$ of tangent directions at x .

Definition 4.1.7. Let S be a surface. By an *embedding* of a graph Γ into S we mean an injective, continuous map $\gamma : |\Gamma| \rightarrow S$ such that

- $\gamma(|\Gamma|^\circ)$ is contained in the interior of S , and $\gamma(\partial|\Gamma|) \subset \partial S$.
- γ is smooth along every edge of $|\Gamma|$.
- For every vertex x of $|\Gamma|$ (corresponding to a vertex or an edge of Γ), the tangent directions of the edge germs on S which leave $\gamma(x)$, are distinct.

Proposition 4.1.8. Let S be a surface with a \mathbb{G} -structure. An embedding of a graph Γ into S endows Γ with a $\Delta\mathfrak{G}$ -structure.

Proof. Let h be a half-edge of Γ incident to the vertex v and with corresponding edge $e = \{h, h'\}$. We denote by $\gamma(|h|)$ the corresponding path in S which runs from $\gamma(v)$ to the midpoint $m(e) = \gamma(e)$.

A \mathbb{G} -structure on S gives, via the homotopy equivalence $\mathbb{G} \rightarrow \text{Homeo}^G(S^1)$, a $\text{Homeo}^G(S^1)$ -structure on each circle $C(T_x S)$. Therefore, by Corollary 2.1.18, each finite subset of $C(T_x S)$ becomes, canonically, a $\Delta\mathfrak{G}$ -structured set. In particular, for each vertex $v \in V$, we have a canonical embedding of sets $H(v) \hookrightarrow C(T_{\gamma(v)} S)$ given by association to a half-edge h the direction given by the germ of the path $\gamma(|h|)$ leaving v . By the above, this provides a canonical $\Delta\mathfrak{G}$ -structure on the set $H(v)$. Further, let e be an edge of Γ given by a pair of half-edges h, h' as above. The two tangent directions to $|\Gamma|$ at $\gamma(m(e))$ correspond to the two half-edges h and h' of e . As before, the embedding $|\Gamma| \rightarrow S$ gives a $\Delta\mathfrak{G}$ -structure on the 2-element set of these directions, i.e., on $\{h, h'\}$. This defines the values of the functor $\tilde{I}_\Gamma : I(\Gamma) \rightarrow \mathcal{G}$ on objects.

To define \tilde{I}_Γ on morphisms, consider one of the half-edges of e , say h , and let $v = s(h)$ be the corresponding vertex. Consider the $\text{Homeo}^G(S^1)$ -structured circles

$$C_v = C(T_{\gamma(v)} S), \quad C_e = C(T_{\gamma(m(e))} S).$$

Let $J_c \subset C_v$ be the union of small non-intersecting closed arcs centered around images of elements of $H(v)$. We write $J_v = A \sqcup B$, where B is the arc corresponding to $h \in H(v)$, and A is the union of all the other arcs, see Figure 2.

Similarly, let $J_e = A' \sqcup B' \subset C_e$ where A' is a small closed arc around the image of h in C_e (denoted a') and B' is a small closed arc around the image of h' in C_e (denoted b'). Note that the orientation torsors of C_v and C_e are identified by parallel transport along $\gamma(|h|)$. We now consider the space Φ formed by all orientation preserving homeomorphisms

$$\phi : (C_v, J_v) \longrightarrow (C_e, J_e) \quad \text{s.t.} \quad \phi(A) \subset A', \phi(B) \subset B'.$$

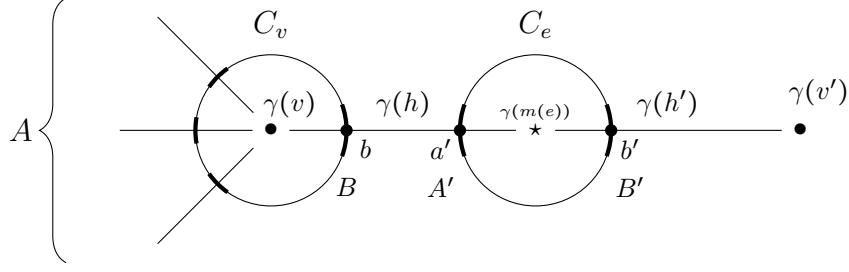


Figure 2: $\Delta\mathcal{G}$ -structure on an embedded graph.

These requirements define ϕ uniquely up to an isotopy (i.e., Φ is connected). Further, by Proposition 2.1.14, Φ is contractible. Consider now the unramified covering

$$\pi : \text{Homeo}^G(C_v, C_e) \longrightarrow \text{Homeo}(C_v, C_e).$$

As Φ is contractible, $\pi^{-1}(\Phi)$ is a disjoint union of components homeomorphic to Φ . We now show that the \mathbb{G} -structure on S defines a canonical choice of a component $\tilde{\Phi} \subset \pi^{-1}(\Phi)$. By Theorem 2.1.16, $\tilde{\Phi}$ will define a structure morphism $H(v) \rightarrow \{e, e'\}$ in \mathcal{G} .

Choose a Riemannian metric on S (a contractible choice). This reduces the structure group of S from \mathbb{G} to G . Let $T : C_v \rightarrow C_e$ be the Riemannian parallel transport along $\gamma(|h|)$. It lifts canonically to a morphism of $\text{Homeo}^G(S^1)$ -structured circles. Denote this structured morphism by \tilde{T} . Let Π be the space of paths $(T_s)_{s \in [0,1]}$ in $\text{Homeo}(C_v, C_e)$ starting at T and ending somewhere in Φ . Any path $(T_s) \in \Pi$ defines a lift $\tilde{\Phi}$ by transporting \tilde{T} along this path. Clearly, this $\tilde{\Phi}$ depends only on the image of (T_s) in $\pi_0(\Pi)$. Note that $\pi_0(\Pi)$ is a torsor over $\mathbb{Z} = \pi_1 \text{Homeo}^+(C_v, C_e)$. Note that T sends the midpoint $b \in B$ (i.e., the image of h in C_v) into the midpoint $b' \in B'$ (i.e., the image of h' in C_e). Therefore we have a distinguished component Π_0 of Π containing paths (T_s) such that $T_s(B)$ meets B' for all s . The transport of \tilde{T} along any path from Π_0 gives the component $\tilde{\Phi}$. This concludes the argument. \square

Examples 4.1.9. (1) Suppose $\Delta\mathcal{G} = \Lambda$. If Γ be embedded into an oriented surface S , then it is classical that Γ is canonically a ribbon graph. Explicitly, each $H(v)$ has a cyclic order from the embedding into the oriented circle $C(T_{\gamma(v)}S)$, while the set of half-edges of any edge has a unique cyclic order since it has cardinality 2.

(2) Suppose $\Delta\mathcal{G} = \Xi$. Let Γ be embedded into an unoriented surface S . Then Γ naturally admits the structure of a Möbius graph. The set $H(v)$ of half-edges incident to v inherits a dihedral order from the embedding into the circle $C(T_{\gamma(v)}S)$. The set $\{h, h'\}$ of half-edges of an edge e is made into a dihedral set by looking at the orientation cover $\varpi : \tilde{\Sigma}S$: we define the orientation torsor $O(\{h, h'\})$ of this set to be the set of sections of π over the image, under γ , of the interior of $|e|$. Finally, we have to provide a lift of the incidence map $i : H(s(h)) \rightarrow \{h, h'\}$ to a morphism of dihedrally ordered sets. Note that, by the path lifting property of the orientation

cover π , the orientation torsors of $H(s(h))$ and $\{h, h'\}$ are canonically identified. For every choice of orientation, there is now a unique linear order on $i^{-1}(h')$ compatible with the corresponding cyclic order of $H(s(h))$. These two linear orders are opposite providing i with the structure of a morphism in Ξ .

- (3) Let $\Delta\mathfrak{G} = \Lambda_\infty$. Assume that Γ is embedded into a framed surface S . This means, in particular, that S is oriented so that we have a principal $GL^+(2, \mathbb{R})$ -bundle Fr_S^+ of positive frames on S . Further, we are given a principal $\widetilde{GL}^+(2, \mathbb{R})$ -bundle $\widetilde{\text{Fr}}_S^+$ covering Fr_S^+ . Thus, each fiber of $\widetilde{\text{Fr}}_S^+$ is a universal cover of the corresponding fiber of Fr_S^+ . Let

$$B^+ = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL^+(2, \mathbb{R}) \mid a, c > 0 \right\}.$$

Then $B^+ \backslash GL^+(2, \mathbb{R}) = S^1$ is the circle of directions of \mathbb{R}^2 . Therefore $B^+ \backslash \text{Fr}_S^+ = C(T_S)$ is the circle bundle of directions of T_S . Further, B^+ , being contractible, lifts canonically to a subgroup $\widetilde{B}^+ \subset \widetilde{GL}^+(2, \mathbb{R})$. The quotient $\widetilde{B}^+ \backslash \widetilde{GL}^+(2, \mathbb{R}) \simeq \mathbb{R}$ is the universal covering of S^1 . So each fiber $\widetilde{C}(T_x S) = \widetilde{B}^+ \backslash \widetilde{\text{Fr}}_{S,x}^+$ is an oriented real line, equipped with the projection $\pi_x : \widetilde{C}(T_x S) \rightarrow C(T_x S)$ exhibiting it as a universal covering of $C(T_x S)$. Therefore, for each vertex $v \in \Gamma$ the preimage of $H(v) \subset C(T_{\gamma(v)} S)$ is naturally a \mathbb{Z} -torsor, thus giving a paracyclic order on $H(v)$. Similarly for each edge $\{h, h'\}$. We leave remaining details to the reader.

- (4) Assume that Γ is embedded into a surface S which has an N -spin structure. Then, similarly to (3), we consider an N -fold covering $\pi_x : \widetilde{C}_N(T_x S) \rightarrow C(T_x S)$ for each $x \in S$. Taking $\pi_{\gamma(v)}^{-1}(H(v))$, we make each $H(v)$ into an N -cyclic set, and similarly for each set $\{h, h'\}$.

4.2 Structured graphs and mapping class groups of structured surfaces

Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group, G its corresponding planar Lie group, \mathbb{G} its thick variant, and let \mathcal{G} denote the category of $\Delta\mathfrak{G}$ -structured sets.

Definition 4.2.1. Let Γ, Γ' be graphs. A *weak equivalence* between Γ and Γ' is a functor $\varphi : I(\Gamma) \rightarrow I(\Gamma')$ of incidence categories such that

- (1) φ induces a bijection of the sets of vertices of valency 1,
- (2) the induced map $|\varphi| : |I(\Gamma)| \rightarrow |I(\Gamma')|$ is homotopy equivalence.

Given a weak equivalence $\varphi : \Gamma \rightarrow \Gamma'$ of graphs we obtain an induced natural transformation $\varphi^* I_{\Gamma'} \rightarrow I_\Gamma$ of incidence diagrams as follows: Given an object x of $I(\Gamma)$ corresponding to a vertex of Γ , we may represent each halfedge e incident to $\varphi(x)$ by a path α in $|I(\Gamma')|$ which starts at $\varphi(x)$ and parametrizes the edge corresponding to e . The germ

of the pullback of α at x determines a unique halfedge at x . This construction canonically determines a map $I_{\Gamma'}(\varphi(x)) \rightarrow I_{\Gamma}(x)$. By a similar construction, one obtains maps $I_{\Gamma'}(\varphi(y)) \rightarrow I_{\Gamma}(y)$ for every object y corresponding to an edge of Γ which assemble to the desired natural transformation.

Definition 4.2.2. Let Γ, Γ' be $\Delta\mathfrak{G}$ -structured graphs. A *weak equivalence* $\varphi : \Gamma \rightarrow \Gamma'$ of $\Delta\mathfrak{G}$ -structured graphs is a weak equivalence of underlying graphs together with a lift of the pullback morphism $\varphi^* I_{\Gamma'} \rightarrow I_{\Gamma}$ to a morphism of \mathcal{G} -diagrams $\varphi^* \widetilde{I}_{\Gamma'} \rightarrow \widetilde{I}_{\Gamma}$. We define $\Delta\mathfrak{G} - \text{Graph}$ to be the category with objects given by (compact) $\Delta\mathfrak{G}$ -structured graphs without vertices of valency 2 and morphisms given by weak equivalences.

Note that this definition is analogous to that of a morphism of ringed spaces. Sometimes we will refer to weak equivalences of structured graphs as *contractions*.

Theorem 4.2.3. *The topological realization $|N(\Delta\mathfrak{G} - \text{Graph})|$ is a classifying space of \mathbb{G} -structured surfaces so that we have a homotopy equivalence*

$$|\Delta\mathfrak{G} - \text{Graph}| \simeq \coprod_{(S,M)} B \text{Mod}^{\mathbb{G}}(S, M),$$

where the coproduct is taken over all topological types of stable marked \mathbb{G} -structured surfaces (S, M) .

The proof will be given in §4.4.B..

Definition 4.2.4. Let (S, M) be a stable marked surface. A *spanning graph* for (S, M) is a graph Γ together with an embedding $\gamma : |\Gamma| \rightarrow S$ such that:

- (1) γ is a homotopy equivalence.
- (2) The restriction of γ defines a homotopy equivalence between $\partial|\Gamma|$ (a finite set of points) and $\partial S - M$ (a finite set of open intervals).

We define the category $P(S, M)$ whose objects are isotopy classes of spanning graphs for (S, M) and with a unique morphism $[\Gamma] \rightarrow [\Gamma']$ if $[\Gamma']$ is obtained from $[\Gamma]$ by collapsing a forest. Note that $P(S, M)$ is in fact a poset.

Theorem 4.2.5. *For any stable marked surface (S, M) the poset $P(S, M)$ is contractible.*

The proof will be given in §I.3.

Assume that (S, M) carries a \mathbb{G} -structure. We then have a functor

$$(4.2.6) \quad \rho : P(S, M) \longrightarrow \Delta\mathfrak{G} - \text{Graph}$$

which chooses for a graph (Γ, γ) in every isotopy class and associates to it the \mathcal{G} -structure on Γ induced from the embedding γ (Proposition 4.1.8). Further, the structured mapping class group $\text{Mod}^{\mathbb{G}}(S, M)$ acts on $P(S, M)$ and we can form the semidirect product

$$\text{Mod}^{\mathbb{G}}(S, M) \ltimes P(S, M)$$

as defined in Definition 1.3.4. The proof of Theorem 4.2.3 amounts to the statement that the functor ρ extends to an equivalence of categories

$$\tilde{\rho}: \text{Mod}^{\mathbb{G}}(S, M) \ltimes P(S, M) \longrightarrow \Delta\mathfrak{G} - \text{Graph},$$

see Theorem 4.4.5.

4.3 Augmented structured trees and operads

The definition of a compact graph given in §4.1 includes 0-valent vertices which can be interpreted as endpoints of external halfedges. This concept is suitable for studying a fixed structured marked surface with boundary. However, when interpreting such structured surfaces as bordisms which can be composed, we need to include combinatorial analogues of collar neighborhoods near the boundary. We realize this by using noncompact structured graphs equipped with an augmentation map for every external halfedge.

Definition 4.3.1. An *augmented $\Delta\mathfrak{G}$ -structured graph* is a possibly noncompact $\Delta\mathfrak{G}$ -structured graph Γ equipped with

- for every external halfedge $e \in H(v)$, an *augmentation map* $\varphi_e : H(v) \rightarrow [1]$ of $\Delta\mathfrak{G}$ -structured sets, satisfying $\varphi_e^{-1}(\varphi(e)) = \{e\}$,

such that every vertex of Γ has valency ≥ 2 .

Given an augmented $\Delta\mathfrak{G}$ -structured graph Γ , we obtain a partition of the set of external halfedges into *incoming* ($\varphi_e(e) = 1$) and *outgoing* ($\varphi_e(e) = 0$). We typically enlarge the incidence diagram $I(\Gamma) \rightarrow \mathcal{G}$ of Γ , also including, for every external halfedge e , the corresponding morphism φ_e . A *contraction* of augmented $\Delta\mathfrak{G}$ -structured graphs is a contraction of the underlying $\Delta\mathfrak{G}$ -structured graphs which commutes with the augmentations.

4.3.A. Augmented structured intervals

A noncompact *interval* is a connected graph with two external halfedges and all vertices of valency 2:

$$\text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---}$$

Consider an augmented $\Delta\mathfrak{G}$ -structured interval Γ with one vertex v , $H(v) = \{e, f\}$ and incidence diagram given by

$$(4.3.2) \quad [1] \xleftarrow{\varphi_e} H(v) \xrightarrow{\varphi_f} [1]$$

where e is incoming and f is outgoing. Given another augmented structured interval

$$[1] \xleftarrow{\varphi_{e'}} H(v') \xrightarrow{\varphi_{f'}} [1]$$

with e' incoming and f' outgoing, we can concatenate with (4.3.2) to obtain the augmented structured interval

$$[1] \xleftarrow{\varphi_e} H(v) \xrightarrow{\varphi_f} [1] \xleftarrow{\varphi_{e'}} H(v') \xrightarrow{\varphi_{f'}} [1]$$

with one internal edge formed by $\{f, e'\} \cong [1]$. Further, there is a contraction from this interval to the interval

$$[1] \xleftarrow{\varphi_e} H(v) \xrightarrow{\varphi_{f'} \varphi_{e'}^{-1} \varphi_f} [1].$$

We define $\text{Tr}_{\Delta\mathfrak{G}}(1)$ to be the category of augmented $\Delta\mathfrak{G}$ -structured intervals with external halfedges labelled by 0 (outgoing) and 1 (incoming) and with contractions as morphisms. We denote by $\pi_0 \text{Tr}_{\Delta\mathfrak{G}}(1)$ the set of isomorphism classes in the groupoid completion of $\text{Tr}_{\Delta\mathfrak{G}}(1)$ so that two objects Γ, Γ' are in the same class if and only if there exists a chain of zigzag contractions

$$\Gamma \longleftarrow \Gamma_1 \longrightarrow \Gamma_2 \longleftarrow \dots \longrightarrow \Gamma'.$$

The above concatenation operation endows the set $\pi_0 \text{Tr}_{\Delta\mathfrak{G}}(1)$ with the structure of a monoid.

Proposition 4.3.3. *The monoid $\pi_0 \text{Tr}_{\Delta\mathfrak{G}}(1)$ can be canonically identified with the group \mathfrak{G}_1^0 .*

Proof. Given an augmented $\Delta\mathfrak{G}$ -structured interval as in (4.3.2), note that φ_e and φ_f are isomorphisms in \mathcal{G} . Using φ_e to identify $H(v)$ with $[1]$, we obtain that the augmented $\Delta\mathfrak{G}$ -structured interval is isomorphic to

$$[1] \xleftarrow{\text{id}} [1] \xrightarrow{g} [1]$$

where g is an automorphism of $[1]$ which induces the identity on the underlying set $\{0, 1\}$. Therefore, we have $g \in \mathfrak{G}_1^0$ where \mathfrak{G}_1^0 denotes the kernel of the map $\lambda : \mathfrak{G}_1 \rightarrow S_2$ from Proposition 1.1.7. It is immediate to verify that this identification is compatible with the group law on \mathfrak{G}_1^0 . \square

Since, for a planar crossed simplicial groups $\Delta\mathfrak{G}$, we have $\mathfrak{G}_1^0 \cong \mathfrak{G}_0$, we obtain the following result.

Corollary 4.3.4. *For a planar crossed simplicial group, the monoid $\pi_0 \text{Tr}_{\Delta\mathfrak{G}}(1)$ is canonically isomorphic to the group \mathfrak{G}_0 .*

4.3.B. Augmented structured trees

The relation between twisted \mathfrak{G}_0 -actions on categories or algebras on one side, and $\Delta\mathfrak{G}$ -structured nerves or Hochschild complexes on the other, can be given a more conceptual explanation. We assume familiarity with the language of operads, referring to [47] for general background.

Let $\text{Tr}_{\Delta\mathfrak{G}}(n)$ denote the category of augmented $\Delta\mathfrak{G}$ -structured trees with external halfedges labelled by $0, 1, \dots, n$ such that the halfedge 0 is outgoing and all remaining halfedges are incoming. The morphisms in $\text{Tr}_{\Delta\mathfrak{G}}(n)$ are given by contractions of augmented $\Delta\mathfrak{G}$ -structured graphs. We define $P_{\Delta\mathfrak{G}}(n)$ to be the set $\pi_0 \text{Tr}_{\Delta\mathfrak{G}}(n)$ of isomorphism classes in the groupoid completion of $\text{Tr}_{\Delta\mathfrak{G}}(n)$. Note that $P_{\Delta\mathfrak{G}}(1)$ is the monoid of $\Delta\mathfrak{G}$ -structured intervals from §4.3.A.. Given augmented $\Delta\mathfrak{G}$ -structured trees Γ, Γ' , an outgoing halfedge $e \in H(v)$ of Γ , and an incoming halfedge $e' \in H(v')$ of Γ' , we obtain a canonical diagram of $\Delta\mathfrak{G}$ -structured sets

$$H(v) \xrightarrow{\varphi_e} [1] = \{e, e'\} = [1] \xleftarrow{\varphi_{e'}} H(v')$$

so that we can concatenate Γ and Γ' forming an augmented $\Delta\mathfrak{G}$ -structured tree with internal edge $\{e, e'\}$. Similarly, given an augmented $\Delta\mathfrak{G}$ -structured tree Γ , an outgoing (resp. incoming) halfedge e , and $g \in P_{\Delta\mathfrak{G}}(1) = \mathfrak{G}_1^0$, we can postcompose the augmentation φ_e with g (resp. g^{-1}). This equips the family of sets $\{P_{\Delta\mathfrak{G}}(n)\}_{n \geq 1}$ with operations

$$m_{a_1, \dots, a_n} : P_{\Delta\mathfrak{G}}(n) \times P_{\Delta\mathfrak{G}}(a_1) \times \dots \times P_{\Delta\mathfrak{G}}(a_n) \longrightarrow P_{\Delta\mathfrak{G}}(a_1 + \dots + a_n), \quad n, a_1, \dots, a_n \geq 1.$$

Proposition 4.3.5. *Let $\Delta\mathfrak{G}$ be a crossed simplicial group.*

- (1) *The action of S_n on $P_{\mathcal{G}}(n)$ by relabelling the incoming external halfedges and the maps $\{m_{a_1, \dots, a_n}\}$ make $P_{\Delta\mathfrak{G}} = \{P_{\Delta\mathfrak{G}}(n)\}_{n \geq 1}$ an operad in the category of sets.*
- (2) *Assume that $\Delta\mathfrak{G}$ is planar. Then algebras over $P_{\Delta\mathfrak{G}}$ are precisely monoids with a twisted action of \mathfrak{G}_0 in the sense of §1.3.*

Proof. Part (1) is clear. We show (2). Given an augmented $\Delta\mathfrak{G}$ -structured tree Γ in $\text{Tr}_{\Delta\mathfrak{G}}(n)$ with augmented incidence diagram $I(\Gamma) \rightarrow \mathcal{G}$, we first argue that we can contract all internal edges to obtain a corolla. To this end, we note that, given an internal edge $\{e, e'\}$ incident to vertices v and v' , the corresponding diagram

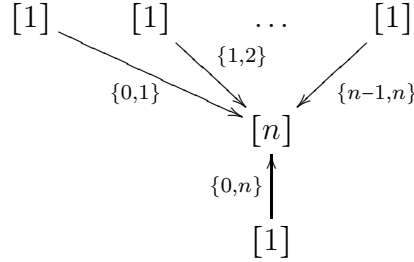
$$H(v) \longrightarrow \{e, e'\} \longleftarrow H(v')$$

has a limit in \mathcal{G} whose underlying set is given by $(H(v) \setminus \{e\}) \cup (H(v') \setminus \{e'\})$. This holds for all planar crossed simplicial groups and can be verified using the topological model of $\Delta\mathfrak{G}$ from §2.1.B.. We can use the cone diagram of this limit in \mathcal{G} to produce a morphism of $\Delta\mathfrak{G}$ -structured graphs $\Gamma \rightarrow \Gamma'$ which collapses the edge $\{e, e'\}$. After collapsing all internal edges of Γ , we obtain an augmented $\Delta\mathfrak{G}$ -structured n -corolla Γ_0 . Using the labels we can identify the set $H(v)$ of halfedges incident to the single vertex of Γ_0 with $\{0, 1, \dots, n\}$. We can find an isomorphism of $\Delta\mathfrak{G}$ -structured sets $f : (\{0, 1, \dots, n\}, \mathcal{O}) \cong [n]$ and since, for planar crossed simplicial groups, \mathfrak{G}_n acts (simply) transitively on $\text{Hom}_{\Delta\mathfrak{G}}([0], [n])$ we may assume that f maps 0 to 0. Therefore, after permuting the labels $1, 2, \dots, n$, we may assume that $H(v) = [n]$. Every morphism $[n] \rightarrow [1]$ such that $\varphi^{-1}(0) = \{0\}$ (corresponding to the outgoing halfedge labeled 0) can be written as a composite φg where $\varphi : [n] \rightarrow [1]$ is the morphism in Δ which maps 0 to 0 and all remaining elements to 1,

and $g \in \text{Stab}(0) \subset \mathfrak{G}_n$. Therefore, we can assume that the augmentation $H(v) = [n] \rightarrow [1]$ of the single outgoing halfedge 0 is given by φ . For every $1 \leq i \leq n$, the subset of $\text{Hom}_{\Delta\mathfrak{G}}([n], [1])$ given by those morphisms such that $\varphi^{-1}(1) = \{i\}$ forms a torsor under the action of $\mathfrak{G}_1^0 \cong \mathfrak{G}_0$. Note that this action of \mathfrak{G}_0 is precisely the operadic action of the monoid $P_{\Delta\mathfrak{G}}(1)$ on the incoming halfedges. Therefore, the collection of isomorphism classes of labelled augmented $\Delta\mathfrak{G}$ -structured n -corollas forms a torsor under the action of

$$\underbrace{P_{\Delta\mathfrak{G}}(1) \times P_{\Delta\mathfrak{G}}(1) \times \cdots \times P_{\Delta\mathfrak{G}}(1)}_n \times S_n.$$

We will now trivialize this torsor, for every n , by choosing a specific $\Delta\mathfrak{G}$ -structured n -corolla in such a way that these chosen n -corollas are closed under operadic composition (for $n > 1$). To this end, choose a duality functor $D : \Delta\mathfrak{G} \rightarrow \Delta\mathfrak{G}^{\text{op}}$ and consider the diagram



in $\Delta \subset \Delta\mathfrak{G}$ and apply D to obtain the incidence diagram of a labeled augmented $\Delta\mathfrak{G}$ -structured n -corolla which we define to be our chosen trivialization. It is easy to verify that, due to the functoriality of D , the collection of corollas thus obtained is closed under operadic composition. This exhibits a copy of the associative operad Ass in P such that every element of P can be uniquely expressed as an element of Ass precomposed with elements of the monoid $P_{\Delta\mathfrak{G}}(1)$ acting on the incoming halfedges. An explicit computation of the action of $P_{\Delta\mathfrak{G}}(1)$ on the outgoing halfedges of Ass in terms of the actions on the incoming halfedges implies the claim that P parametrizes algebras with twisted \mathfrak{G}_0 -action. \square

Remark 4.3.6. Proposition 4.3.5 can be enhanced by considering the geometric realization of the categories $\text{Tr}_{\Delta\mathfrak{G}}(n)$ instead of the set $P_{\Delta\mathfrak{G}}(n)$ of its connected components. In this way, we obtain an operad $\text{NTr}_{\Delta\mathfrak{G}}$ in the category of topological spaces which describes A_∞ -monoids with a twisted coherent action of \mathfrak{G}_0 .

Remark 4.3.7. The results of this section suggest that augmented $\Delta\mathfrak{G}$ -structured *graphs* can be interpreted in modular operadic terms generalizing the well-known approaches to 2-dimensional topological field theories in the cyclic and dihedral cases (cf. [16, 10]). We leave the systematic treatment of these generalizations to future work.

4.4 Structured graphs and structured moduli spaces

4.4.A. Orbicell decompositions

By a *d-cell* we mean a topological space σ homeomorphic to an open ball $B^d \subset \mathbb{R}^d$. By a *d-orbicell* we mean a topological groupoid equivalent to $D \backslash \sigma$ where $\sigma \simeq B^d$ is a *d-cell* and D is a discrete group acting on σ via a homomorphism $\phi : D \rightarrow \text{Homeo}(\sigma)$ whose image is identified with a finite group of linear transformations of B^d . Note that we do not require $\text{Ker}(\phi)$ to be finite.

Definition 4.4.1. Let X be a topological groupoid. An *orbicell decomposition* of X is a finite filtration

$$\mathcal{X} = (\mathcal{X}_0 \subset \mathcal{X}_1 \subset \dots \subset \mathcal{X}_n = X)$$

of X by closed topological subgroupoids \mathcal{X}_d such that:

- (1) Each \mathcal{X}_d is a full subcategory in X .
- (2) Each component of $\mathcal{X}_d - \mathcal{X}_{d-1}$ is a disjoint union of *d-orbicells*.

Example 4.4.2. Let Y be a topological space with a cell decomposition \mathcal{Y} . Let D be a discrete group acting on Y , preserving \mathcal{Y} and such that for each cell σ of \mathcal{Y} the stabilizer $\text{Stab}(\sigma)$ acting on σ , makes $\text{Stab}(\sigma) \backslash \sigma$ into an orbicell. Then the groupoid $D \backslash Y$ has an orbicell decomposition into orbicells $\text{Stab}(\sigma) \backslash \sigma$ where σ runs over orbits of D on the set of cells of \mathcal{Y} .

Recall that for ordinary spaces (not groupoids) Y there is a concept of a *regular cell decomposition* which is a cell decomposition \mathcal{Y} such that the closure of each *d-cell* σ of \mathcal{Y} is homeomorphic to a closed ball in \mathbb{R}^d (whose boundary is therefore a $(d-1)$ -sphere with further cell decomposition induced by \mathcal{Y}). We were not able to find in the literature an intrinsic analog of this concept for orbicell decompositions. The following concept will be sufficient for our purpose.

Definition 4.4.3. An orbicell decomposition \mathcal{X} of a topological groupoid X is called *quotient-regular*, if there is an equivalence of X with a quotient groupoid $D \backslash Y$ and a regular D -invariant cell decomposition \mathcal{Y} of Y as in Example 4.4.2, which induces \mathcal{X} .

For a quotient-regular cell decomposition \mathcal{X} we have a category $\text{Cell}(\mathcal{X})$ whose objects are cells of \mathcal{Y} and morphisms are induced by action of D and inclusion of cells. In other words, $\text{Cell}(\mathcal{X})$ is the semi-direct product of D and the poset of cells of \mathcal{Y} . Note that isomorphism classes of objects of $\text{Cell}(\mathcal{X})$ are in bijection with D -orbits on cells of Y , i.e., with distinct orbicells in $\sqcup \mathcal{X}_d - \mathcal{X}_{d-1}$. The following is then straightforward and follows from Proposition I.2.2.

Proposition 4.4.4. *If \mathcal{X} is a quotient-regular orbicell decomposition of X , then the classifying space of the (non-topological) category $\text{Cell}(\mathcal{X})$ is homotopy equivalent to the classifying space of X as a topological category.* \square

Note that such a statement cannot be true without some regularity assumptions already in the case of ordinary (not orbi) cell decompositions as can be seen by considering S^2 decomposed into \mathbb{R}^2 and ∞ .

4.4.B. Decompositions of the structured moduli spaces

We now prove the following fact which, in virtue of Proposition 4.4.4, implies Theorem 4.2.3.

Theorem 4.4.5. *Consider the orbifold*

$$\mathcal{M}^{G_{\text{conf}}} = \coprod_{(S,M)} \mathcal{M}^{G_{\text{conf}}}(S, M)$$

given by the disjoint union over all possible topological types of stable marked surfaces. Then $\mathcal{M}^{G_{\text{conf}}}$ has a quotient-regular orbicell decomposition \mathcal{X} such that we have an equivalence of categories $\text{Cell}(\mathcal{X}) \simeq \Delta\mathfrak{G} - \text{Graph}$.

Proof. (a) Oriented case $G = SO(2)$. In this case, the statement (that the usual moduli space of stable marked *Riemann surfaces* has an orbicell decomposition labelled by ribbon graphs) is well known, see, e.g., [3, 57] and references therein. We indicate here the main steps which we later analyze to deduce the general case.

First, each component of $\mathcal{M}^{SO(2)_{\text{conf}}}$ is a quotient groupoid

$$\mathcal{M}^{SO(2)_{\text{conf}}}(S, M) = \text{Mod}^+(S, M) \backslash \backslash \text{Teich}^+(S, M).$$

Second, $\text{Teich}^+(S, M)$ has a $\text{Mod}^+(S, M)$ -invariant regular cell decomposition (triangulation) constructed by Harer [33] and described in Appendix I.3. Denote this cell decomposition by \mathcal{Y} . The simplices of \mathcal{Y} correspond to tessellations of (S, M) or, dually, to isotopy classes of spanning graphs for (S, M) . Inclusions of simplices correspond to coarsenings of tessellations or, dually, to contractions of spanning graphs. Any spanning graph for (S, M) carries an induced ribbon structure. Therefore, orbicells in the quotient orbicell decomposition, which we denote $\mathcal{X}_{S,M}$, form a full subcategory in $\Lambda - \text{Graph}$ with objects those graphs that appear as spanning graphs for (S, M) . Finally, each ribbon graph Γ give arise to an oriented marked surface (S, M) obtained by “thickening” Γ , so that Γ becomes a spanning graph for (S, M) , see, e.g., [23] §3.3.4. This shows that for $\mathcal{X} = \coprod_{(S,M)} \mathcal{X}_{S,M}$, we have $\text{Cell}(\mathcal{X}) \simeq \Lambda - \text{Graph}$.

Note that Costello [15] introduced another topological groupoid \mathcal{N} with an orbicell decomposition labelled by $\Lambda - \text{Graph}$ which is homotopy equivalent, but not homeomorphic to $\mathcal{M}^{SO(2)_{\text{conf}}}$.

(b) Unoriented case $G = O(2)$. It is treated similarly to the oriented case, using the identification

$$\mathcal{M}^{O(2)_{\text{conf}}}(S, M) = \text{Mod}(S, M) \backslash \backslash \text{Teich}(S, M)$$

and the triangulation \mathcal{Y} of $\text{Teich}(S, M)$ described in §I.3. As before, the simplices of \mathcal{Y} correspond to triangulations of (S, M) and give Möbius graphs as $\text{Mod}(S, M)$ -orbits. Further, any Möbius graph Γ can be “thickened” producing an unoriented surface (S, M) in which it is spanning. So, for the resulting quotient orbicell decomposition \mathcal{X} of $\mathcal{M}^{O(2)\text{conf}}$ we have $\text{Cell}(\mathcal{X}) \simeq \Xi - \text{Graph}$.

Note that an analog of Costello’s approach [15] for Klein surfaces was developed by Braun [10].

(c) General case: G an arbitrary planar Lie group. Depending on whether G preserves orientation or not, we consider one of the two projections

$$\pi : \mathcal{M}^{G\text{conf}} \longrightarrow \mathcal{M}^{SO(2)\text{conf}}, \quad \pi : \mathcal{M}^{G\text{conf}} \longrightarrow \mathcal{M}^{O(2)\text{conf}}$$

obtained by forgetting the extra structure. In each case, the projection π is an unramified covering of orbifolds, so the preimage of any orbicell in the target is a disjoint union of orbicells in the source. We thus obtain an orbicell decomposition of $\mathcal{M}^{G\text{conf}}$ which we now analyze. We explain the case when G does not preserve the orientation, the other case being similar but easier. Denote by \mathcal{X} the orbicell decomposition of $\mathcal{M}^{O(2)\text{conf}}$ and by \mathcal{X}^G the preimage decomposition of $\mathcal{M}^{G\text{conf}}$. Since any G_{conf} -structured marked surface is \mathbb{G} -structured, any tessellation \mathcal{P} of the surface makes the dual graph of \mathcal{P} into a $\Delta\mathbb{G}$ -structured graph. We obtain therefore a commutative diagram of categories

$$\begin{array}{ccc} \text{Cell}(\mathcal{X}^G) & \xrightarrow{\pi_{\text{Cell}}} & \text{Cell}(\mathcal{X}) \\ q_G \downarrow & & \downarrow q \\ \Delta\mathbb{G} - \text{Graph} & \xrightarrow{\varpi} & \Xi - \text{Graph} \end{array}$$

where q is an equivalence by the case (b) above. We prove that q_G is an equivalence as well.

For this, we show that for each orbicell $\sigma \in \text{Cell}(\mathcal{X})$ the functor induced by q_G on fibers (comma-categories)

$$q_G/\sigma : \pi_{\text{Cell}}/\sigma \longrightarrow \omega/p(\sigma)$$

is an equivalence of categories. We note that π_{Cell}/σ is identified with π/X_σ , the fiber of π over the Klein surface $(X_\sigma, M) \in \mathcal{M}^{O(2)\text{conf}}$ corresponding to the barycenter of the (orbi)simplex σ . On the other hand, by Proposition 3.2.3, for any $(X, M) \in \mathcal{M}^{O(2)\text{conf}}$, the fiber $\pi/(X, M)$ is identified with the groupoid $\mathbb{G}\text{Str}(X)$ of \mathbb{G} -structures on X . For a Möbius graph Γ , we denote by $\Delta\mathbb{G}\text{Str}(\Gamma)$ the fiber of ϖ over Γ . In other words, this is the category (groupoid) of $\Delta\mathbb{G}$ -structures on Γ compatible with the given Möbius structure and morphisms of such structures identical on Γ . We conclude the proof using the following lemma. \square

Lemma 4.4.6. *Let (S, M) be a stable marked surface and Γ be a spanning graph for (S, M) . Then the functor*

$$q_\Gamma : \mathbb{G}\text{Str}(S - (M \cap S^\circ)) \rightarrow \Delta\mathbb{G}\text{Str}(\Gamma)$$

is an equivalence of categories.

Proof. We first show that q_Γ is essentially surjective. Let F_Γ be a $\Delta\mathfrak{G}$ -structure on Γ . By Theorem 2.1.16, F_Γ can be seen as a datum, for each $x \in \Gamma \subset S'$, of a $\text{Homeo}^G(S^1)$ -structure on the circle of directions $C_x(S)$. Moreover, these structures are compatible as x moves in Γ , thus providing a $\text{Homeo}^G(S^1)$ -structure on $C(T_S)|_\Gamma$, the restriction to Γ of the circle bundle $C(T_S)$. Since Γ is homotopy equivalent to $S - M$ and hence to $S - (M \cap S^\circ)$, we get a unique, up to isomorphism, $\text{Homeo}^G(S^1)$ -structure on $C(T_{S-(M \cap S^\circ)})$. Finally, since $\mathbb{G} \rightarrow \text{Homeo}^G(S^1)$ is a homotopy equivalence, we obtain from this a \mathbb{G} -structure on $S - (M \cap S^\circ)$. Thus q_Γ is essentially surjective.

Second, we show that q_Γ induces bijections on Hom-sets. Because $\mathbb{G}\text{Str}(S)$ is a groupoid, it is enough to show that for each \mathbb{G} -structure F on $S - (M \cap S^\circ)$ the induced map of automorphism groups

$$q_* : \text{Aut}(F) \longrightarrow \text{Aut}(F_\Gamma)$$

is an isomorphism. Note that q_* actually comes (by passing to global sections) from a homomorphism of sheaves of groups on Γ formed by local isomorphisms:

$$\underline{q}_* : \underline{\text{Aut}}(F) \longrightarrow \underline{\text{Aut}}(F_\Gamma).$$

So it is enough to prove that \underline{q}_* is an isomorphism of sheaves. By Proposition 3.1.5(b1), $\underline{\text{Aut}}(F) = \underline{K}^{\text{or}}$ is the local system obtained by the orientation twist of $K = \text{Ker}\{\mathbb{G} \rightarrow GL(2, \mathbb{R})\}$. Now note that K can also be identified with $\text{Ker}\{\mathfrak{G}_n \rightarrow D_{n+1}\}$ for any n by Theorem 1.5.6(b2). An automorphism of the $\Delta\mathfrak{G}$ -structure F_Γ (identical on Γ) is therefore a section of a local system \mathcal{L} on Γ whose stalk at each point is isomorphic to K . The final identification of \mathcal{L} with $\underline{K}^{\text{or}}$ and of q_* with the identity of $\underline{K}^{\text{or}}$ is left to the reader. \square

5 2-Segal $\Delta\mathcal{G}$ -objects and invariants of \mathbb{G} -structured surfaces

5.1 The 1- and 2-Segal conditions

Let Δ be the *simplex category* of finite nonempty linearly ordered sets of the form $[n] = \{0, 1, \dots, n\}$ with morphisms given by monotone maps. Given categories I and \mathbf{C} , we introduce the notation

$$\mathbf{C}^I = \text{Fun}(I, \mathbf{C}), \quad \mathbf{C}_I = \text{Fun}(I^{\text{op}}, \mathbf{C})$$

for the categories of covariant and contravariant functors. The objects of \mathbf{C}_Δ are called *simplicial objects* in \mathbf{C} . If \mathbf{C} admits limits, then we can use the formalism of Kan extension, to enlarge the domain of definition of a simplicial object from Δ to Set_Δ . More precisely, let

$$\Upsilon : \Delta^{\text{op}} \rightarrow (\text{Set}_\Delta)^{\text{op}}$$

be the opposite of the Yoneda embedding. Then there exists an adjunction

$$\Upsilon^* : \mathbf{C}_{\text{Set}_\Delta} \longleftrightarrow \mathbf{C}_\Delta : \Upsilon_*$$

where, for a simplicial set K and a simplicial object X , we have the explicit formula

$$\Upsilon_* X(K) = \lim_{\longleftarrow \{\Delta^n \rightarrow K\}}^{\mathbf{C}} X_n$$

given by a limit in \mathbf{C} over the category of simplices of K . We introduce the notation $(K, X) := \Upsilon_* X(K)$ and refer to it as the *object of K -membranes in X* . Assume further, that \mathbf{C} is equipped with a combinatorial model structure. Then we can derive the above adjunction to obtain

$$L\Upsilon^* : \text{Ho}(\mathbf{C}_{\text{Set}_\Delta}) \longleftrightarrow \text{Ho}(\mathbf{C}_\Delta) : R\Upsilon_*$$

where, as above, we have an explicit formula

$$R\Upsilon_* X(K) \cong \text{holim}_{\longleftarrow \{\Delta^n \rightarrow K\}}^{\mathbf{C}} X_n$$

in terms of a homotopy limit in \mathbf{C} . We call $(K, X)_R := R\Upsilon_* X(K)$ the *derived object of K -membranes in X* .

We will be particularly interested in membranes parametrized by the simplicial sets introduced in the following examples.

Example 5.1.1. For $n \geq 1$, we denote by $I[n] \subset \Delta^n$ the simplicial subset corresponding to the subdivided interval

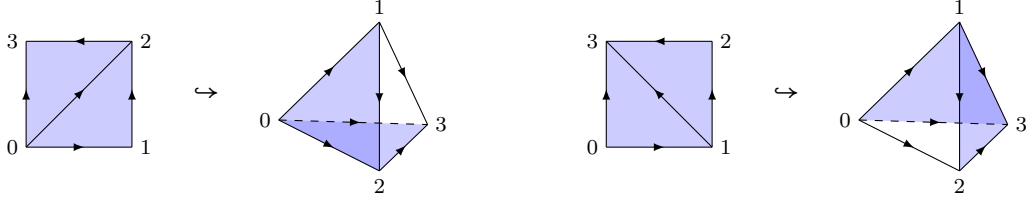
$$\bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet \quad \dots \quad \bullet \xrightarrow{\quad} \bullet$$

$0 \qquad 1 \qquad 2 \qquad \dots \qquad n-1 \qquad n$

which can be more formally described as the pushout of simplicial sets

$$\Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \dots \amalg_{\Delta^0} \Delta^1.$$

Example 5.1.2. Let P_{n+1} be a convex plane $(n+1)$ -gon with the set of vertices labelled counterclockwise by $M = \{0, 1, \dots, n\}$. Let \mathcal{T} be any triangulation of P_{n+1} with vertex set M . A geometric triangle σ of \mathcal{T} with vertices i, j, k gives rise to a unique full simplicial subset $\Delta^2 \subset \Delta^n$ with vertices $\{i\}, \{j\}, \{k\}$. The union, or more formally the pushout, of the simplicial subsets of Δ^n corresponding to all triangles of \mathcal{T} defines a simplicial subset $\Delta^\mathcal{T} \subset \Delta^n$ whose geometric realization is homeomorphic to P_{n+1} . For example, the two simplicial subsets of Δ^3 corresponding to the two triangulations of a planar square can be visualized as follows:



We recall the following definitions from [22], the first one being a variant of Rezk's Segal condition [?].

Definition 5.1.3. Let \mathbf{C} be a combinatorial model category, and let $X \in \mathbf{C}_\Delta$ be a simplicial object.

- (1) We say that X is *1-Segal* if, for every $n \geq 1$, the morphism

$$f_n : X_n \longrightarrow (I[n], X)_R = X_1 \times_{X_0}^R X_1 \times_{X_0}^R \dots \times_{X_0}^R X_1,$$

induced by the embedding $I[n] \hookrightarrow \Delta^n$, is a weak equivalence in \mathbf{C} .

- (2) We say that X is *2-Segal* if, for every $n \geq 2$ and every triangulation \mathcal{T} of P_{n+1} , the morphism

$$f_\mathcal{T} : X_n \longrightarrow (\Delta^\mathcal{T}, X)_R,$$

induced by the embedding $\Delta^\mathcal{T} \hookrightarrow \Delta^n$, is a weak equivalence in \mathbf{C} .

Remark 5.1.4. Let \mathbf{C} be an ordinary category with limits and colimits. We can equip \mathbf{C} with the trivial model structure such that the weak equivalences are given by the class of isomorphisms. The above Segal conditions then involve underived membrane spaces require the corresponding morphisms f_n , respectively $f_\mathcal{T}$, to be isomorphisms in \mathbf{C} .

5.2 Categorized structured state sums

Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group. Then the category \mathcal{G} of $\Delta\mathfrak{G}$ -structured sets comes equipped with an interstice duality functor $D : \mathcal{G} \rightarrow \mathcal{G}^{\text{op}}$. Let Γ be a $\Delta\mathfrak{G}$ -structured graph with $\Delta\mathfrak{G}$ -structure $I(\Gamma) \rightarrow \mathcal{G}$. Postcomposing with D and the opposite of the Yoneda embedding

$$\Upsilon : \mathcal{G}^{\text{op}} \longrightarrow (\text{Set}_{\mathcal{G}})^{\text{op}} \xrightarrow{|\Delta\mathfrak{G}|} (\text{Set}_{\Delta\mathfrak{G}})^{\text{op}},$$

we obtain a diagram $I(\Gamma) \rightarrow \text{Set}_{\Delta\mathfrak{G}}^{\text{op}}$.

Definition 5.2.1. Given a $\Delta\mathfrak{G}$ -structured graph Γ , the limit of the associated diagram $I(\Gamma) \rightarrow (\mathcal{S}et_{\Delta\mathfrak{G}})^{\text{op}}$, denoted by $\Delta\mathfrak{G}^\Gamma$, is called the $\Delta\mathfrak{G}$ -set realization of Γ .

Proposition 5.2.2. The $\Delta\mathfrak{G}$ -set realization extends to a functor

$$\Delta\mathfrak{G} - \text{Graph} \longrightarrow (\mathcal{S}et_{\Delta\mathfrak{G}})^{\text{op}}.$$

Proof. A morphism φ between $\Delta\mathfrak{G}$ -structured graphs Γ and Γ' consists of a functor $\varphi : I(\Gamma) \rightarrow I(\Gamma')$ of incidence categories together with a natural transformation $\eta : \tilde{I}_{\Gamma'} \circ \varphi \rightarrow \tilde{I}_\Gamma$. We obtain a canonical sequence of morphisms

$$\lim \Upsilon D \tilde{I}_{\Gamma'} \longrightarrow \lim \Upsilon D \tilde{I}_{\Gamma'} \varphi \xrightarrow{\eta} \lim \Upsilon D \tilde{I}_\Gamma$$

whose composite defines the desired morphism $\Delta\mathfrak{G}^{\Gamma'} \rightarrow \Delta\mathfrak{G}^\Gamma$. Here the first morphism is the natural morphism $\lim F \rightarrow \lim F\varphi$ which exists for any functor $F : I(\Gamma') \rightarrow \mathbf{C}$ provided that both limits exist. It is straightforward to verify the functoriality of this association. \square

Let \mathbf{C} be a combinatorial model category, and let $X : \Delta\mathfrak{G}^{\text{op}} \rightarrow \mathbf{C}$ a \mathcal{G} -structured 2-Segal object in \mathbf{C} . We form the right homotopy Kan extension of X along the Yoneda embedding $\Upsilon : \Delta\mathfrak{G}^{\text{op}} \rightarrow (\mathcal{S}et_{\Delta\mathfrak{G}})^{\text{op}}$ and take the composite with the above functors to obtain a canonical functor

$$(5.2.3) \quad RX : P(S, M) \longrightarrow \text{Ho}(\mathbf{C}), \Gamma \mapsto RX(\Gamma)$$

where $RX(\Gamma) := R\Upsilon_* X(\Delta\mathfrak{G}^\Gamma)$ and Γ is equipped with its natural $\Delta\mathfrak{G}$ -structured from Proposition 4.1.8. Finally, we define

$$X(S, M) := \varinjlim RX.$$

Theorem 5.2.4. The object $X(S, M)$ comes equipped with a canonical isomorphism

$$(5.2.5) \quad RX(\Gamma) \xrightarrow{\cong} X(S, M),$$

for every object Γ in $P(S, M)$.

Proof. The claimed isomorphisms are simply the maps which constitute the colimit cone under the diagram RX . These maps are isomorphisms since, by Theorem I.1.4, the diagram RX is indexed by a simply connected index category and, due to the 2-Segal property satisfied by X , maps all morphisms to isomorphisms in $\text{Ho}(\mathbf{C})$ (see [23]). \square

Remark 5.2.6. Theorem 5.2.4 shows that the structured surface invariant $X(S, M)$ can be explicitly computed via the formula

$$X(S, M) \cong \text{holim}_{\longleftarrow \{\Delta\mathfrak{G}^n \rightarrow \Delta\mathfrak{G}^\Gamma\}} X_n$$

where Γ is any chosen spanning graph embedded in S . This formula can be regarded as a categorified variant of the state sum formulas for surface invariants in the context of topological field theory.

Consider the functor

$$\Delta\mathfrak{G} - \text{Graph} \longrightarrow \mathbf{C}, \Gamma \mapsto RX(\Gamma).$$

Due to the 2-Segal property of X , it maps weak equivalences of $\Delta\mathfrak{G}$ -structured graphs to weak equivalence in \mathbf{C} . We can pass to classifying spaces to obtain a map of topological spaces.

$$|\Delta\mathfrak{G} - \text{Graph}| \longrightarrow |W_{\mathbf{C}}|$$

where $W_{\mathbf{C}}$ denotes the category of weak equivalences in \mathbf{C} . By Theorem 4.2.3, we have

$$|\Delta\mathfrak{G} - \text{Graph}| \simeq \coprod_{(S,M)} B \text{Mod}^{\mathbb{G}}(S, M)$$

so that we obtain, for every stable \mathbb{G} -structured surface (S, M) , a map

$$\rho : B \text{Mod}^{\mathbb{G}}(S, M) \longrightarrow |W_{\mathbf{C}}|.$$

The topological space $|W_{\mathbf{C}}|$ is a model for the maximal ∞ -groupoid in the simplicial localization of \mathbf{C} with respect to W (cf. [?]). Therefore, the map ρ realizes a coherent action of the mapping class group $\text{Mod}^{\mathbb{G}}(S, M)$ on $\rho(*) \cong X(S, M)$.

Theorem 5.2.7. *The structured mapping class group $\text{Mod}^{\mathbb{G}}(S, M)$ acts coherently on the object $X(S, M)$.*

5.3 Examples

We sketch some explicit examples of structured 2-Segal objects and describe the corresponding categorified state sum invariants. The examples arise more naturally as cosimplicial objects

$$\mathcal{F} : \Delta \longrightarrow \mathbf{C}$$

with additional structure. We say that \mathcal{F} is 2-coSegal if \mathcal{F}^{op} is 2-Segal and translate all other concepts in a similar way.

5.3.A. Fundamental groupoids

Let **Grpds** denote the category of small groupoids. Given a marked unoriented surface (S, M) , we define the fundamental groupoid $\Pi_1(S, M)$ to be the groupoid with set of objects M and morphisms given by homotopy classes of paths in S between points in M . The category **Grpds** is equipped with the trivial model structure so that weak equivalences are given by isomorphisms (as opposed to equivalences) of groupoids.

Consider the codihedral groupoid

$$\mathcal{F} : \Xi \longrightarrow \mathbf{Grpds}, [n] \mapsto \Pi_1(D, \{0, 1, \dots, n\})$$

where $\Pi_1(D, \{0, 1, \dots, n\})$ denotes the fundamental groupoid of the unit disk D in \mathbb{C} , relative to the subset $\{0, 1, \dots, n\} \subset D$ of $(n+1)$ st roots of unity. The dihedral functoriality is most naturally seen using a topological model for Ξ : We represent a morphism $[m] \rightarrow [n]$ by a homotopy equivalence $S^1 \rightarrow S^1$ which maps $\{0, 1, \dots, m\}$ to $\{0, 1, \dots, n\}$. We attach disks to obtain a map of pairs $(D, \{0, 1, \dots, m\}) \rightarrow (D, \{0, 1, \dots, n\})$ and pass to fundamental groupoids.

Proposition 5.3.1. *The codihedral groupoid \mathcal{F} is 2-coSegal. The value $\mathcal{F}(S, M)$ of \mathcal{F} on an unoriented marked surface (S, M) is the fundamental groupoid $\Pi_1(S, M)$ of S relative to M .*

Proof. To verify the 2-Segal property, we have to show that, for any subdivision of the $n+1$ -gon P_n into two polygons P_m with vertices $\{0, 1, \dots, i, j, j+1, \dots, n\}$ and P_l with vertices $\{i, i+1, \dots, j\}$, the corresponding functor

$$f : \Pi_1(D, \{0, 1, \dots, i, j, j+1, \dots, n\}) \coprod_{\Pi_1(D, \{i, j\})} \Pi_1(D, \{i, i+1, \dots, j\}) \longrightarrow \Pi_1(D, \{0, 1, \dots, n\})$$

is an isomorphism of groupoids. Up to isomorphism of groupoids we can alternatively describe f as the natural functor

$$f' : \Pi_1(P_m) \coprod_{\Pi_1(P_l)} \Pi_1(P_l) \longrightarrow \Pi_1(P_n)$$

induced from expressing P_n as the union of P_m and P_l with intersection $P_m \cap P_l = P_l$. Here, all fundamental groupoids are taken with respect to the vertices of the respective polygon. The fact that f' is an isomorphism of groupoids is an easy instance of van Kampen's theorem (see, e.g., [11, 6.7.2]). Similarly, to compute $\mathcal{F}(S, M)$, note that we can pass to the dual of a spanning graph in $S \setminus M$ to express (S, M) as a union of polygons. Interpreting formula (5.2.5) in these terms, another application of van Kampen's theorem shows that $\mathcal{F}(S, M)$ is the fundamental groupoid $\Pi_1(S, M)$ computed as a colimit parametrized by the above polygonal subdivision of (S, M) . \square

5.3.B. Relative homology

Let \mathbf{A} be the category of abelian groups. For $n \geq 0$, we define the abelian group \mathcal{F}_n to be the kernel of the homomorphism

$$\mathbb{Z}^{n+1} \rightarrow \mathbb{Z}, (z_0, z_1, \dots, z_n) \mapsto \sum z_i.$$

These abelian groups organize into a codihedral abelian group

$$\mathcal{F} : \Xi \longrightarrow \mathbf{A}, [n] \mapsto \mathcal{F}_n.$$

This can be seen by forming a larger codihedral abelian group given as the composite

$$\Xi \xrightarrow{\lambda} \mathcal{S}et \longrightarrow \mathbf{A}$$

and verifying that \mathcal{F} is a codihedral subgroup. Here, λ denotes the functor from Proposition 1.1.7 and $\mathcal{S}et \rightarrow \mathbf{A}$ is the functor of taking the free abelian group on a set.

Proposition 5.3.2. *The codihedral abelian group \mathcal{F} is 2-coSegal. The value $\mathcal{F}(S, M)$ of \mathcal{F} on an unoriented marked surface (S, M) is the relative homology $H_1(S, M)$ of S relative to M .*

Proof. Note that \mathcal{F}_n can be interpreted as the relative homology of the pair $(D, \{0, 1, \dots\})$ from §5.3.A.. The argument is now very similar to the proof of Proposition 5.3.1, using the Mayer-Vietoris sequence instead of van Kampen's theorem. \square

5.3.C. Topological Fukaya categories

Let \mathbf{k} be a field. In [23], we constructed a cocyclic differential $\mathbb{Z}/2\mathbb{Z}$ -graded category

$$\mathcal{F} : \Lambda \longrightarrow \mathrm{dgc}at_{\mathbf{k}}^{(2)}, [n] \mapsto \mathrm{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$$

and verified that \mathcal{F} is 2-coSegal with respect to the Morita model structure on $\mathrm{dgc}at^{(2)}$. Here, $\mathrm{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$ denotes the category of $\mathbb{Z}/(n+1)$ -graded matrix factorizations of the polynomial z^{n+1} . The value of \mathcal{F} on an oriented marked surface (S, M) is called the topological coFukaya category of (S, M) . See [23] for some explicit calculations of $\mathcal{F}(S, M)$.

As a slight variation, there is a coparacyclic differential \mathbb{Z} -graded category

$$\mathcal{F}_{\infty} : \Lambda_{\infty} \longrightarrow \mathrm{dgc}at_{\mathbf{k}}, [n] \mapsto \mathrm{MF}^{\mathbb{Z}}(\mathbf{k}[z], z^{n+1})$$

given by \mathbb{Z} -graded matrix factorizations. The value of \mathcal{F}_{∞} on a framed marked surface (S, M) provides a \mathbb{Z} -graded variant of the topological coFukaya category.

Interpolating between these examples, we can define an N -cocyclic differential $\mathbb{Z}/2N$ -graded category

$$\mathcal{F}_N : \Lambda_N \longrightarrow \mathrm{dgc}at_{\mathbf{k}}^{(2N)}, [n] \mapsto \mathrm{MF}^{\mathbb{Z}/N(n+1)}(\mathbf{k}[z], z^{n+1})$$

so that $\mathcal{F}_N(S, M)$ provides a $\mathbb{Z}/2N$ -graded variant of the topological coFukaya category associated to any marked N -spin surface (S, M) .

5.3.D. Structured nerves

Let $\Delta\mathfrak{G}$ be a planar crossed simplicial group. Consider the functor

$$\mathcal{F} : \Delta\mathfrak{G} \longrightarrow \mathfrak{G}_0 - \mathrm{Cat}, [n] \mapsto \mathrm{FC}(\Delta\mathfrak{G}^n|_{\Delta})$$

from Proposition 1.6.2. The category $\mathfrak{G}_0 - \mathrm{Cat}$ can be equipped with a model structure so that weak equivalences are morphisms which, on the underlying categories, induce equivalences. An argument similar to [22, 7.5] can be used to prove the following statement.

Proposition 5.3.3. *The $\Delta\mathfrak{G}$ -object \mathcal{F} is 2-coSegal.*

The value of \mathcal{F} on a \mathbb{G} -structured surface can be explicitly computed in various examples. The question of finding a general intrinsic interpretation of $\mathcal{F}(S, M)$ in terms of the \mathbb{G} -structured surface is open.

I The tessellation complex and the Teichmüller space

I.1 The Stasheff polytopes and the tessellation complex

Let P be a convex polygon in the plane, with the set of vertices M . A *tessellation* of P is a (possibly empty) set of non-intersecting diagonals in P . The set $\text{Tess}(P)$ of tessellations of P is partially ordered by inclusions of sets of diagonals. A tessellation $\mathcal{P} \in \text{Tess}(P)$ subdivides P into several sub-polygons P' . We write (allowing a slight abuse of notation) $P' \in \mathcal{P}$. Thus maximal elements of $\text{Tess}(P)$ are triangulations: tessellations into triangles.

As well known, there is a convex polytope $\mathbb{T}(P)$ canonically associated to P and called the *Stasheff polytope* of P . The poset of faces of $\mathbb{T}(P)$ is in an anti-isomorphism with $\text{Tess}(P)$. Thus, vertices of $\mathbb{T}(P)$ correspond to triangulations and edges to tessellations into all triangles and one 4-gon. The face of $\mathbb{T}(P)$ corresponding to a tessellation \mathcal{P} , is identified with

$$\mathbb{T}(\mathcal{P}) := \prod_{P' \in \mathcal{P}} \mathbb{T}(P').$$

The combinatorial structure of $\mathbb{T}(P)$ depends only on the combinatorial structure of P , i.e., on the number of vertices. Taking P to be the regular polygon P_{n+1} in $\mathbb{C} = \mathbb{R}^2$ with the set of vertices

$$M = \langle n \rangle = \{ \exp(2\pi i k / (n+1)), k = 0, 1, \dots, n \},$$

we see that the dihedral group D_{n+1} acts on $\mathbb{T}(P)$ by affine automorphisms. This means that we can associate a Stasheff polytope \mathbb{T}_M to any finite dihedral ordinal $M \in \Xi$, $|M| \geq 3$.

More generally, by a (*curvilinear*) *polygon* we will mean a stable marked surface (S, M) such that S is, topologically, a disk and $M \subset \partial S$. In this case M has a canonical dihedral order and we associate to (S, M) the Stasheff polytope $\mathbb{T}(S, M) := \mathbb{T}_M$.

Let now (S, M) be an arbitrary stable marked surface. Following [33, 27], we call an *arc* on (S, M) an isotopy class of unoriented paths γ beginning and ending on M (possibly at the same point) such that:

- (1) γ does not meet any elements of M other than containing them as endpoints.
- (2) γ does not have any self-intersection points except possibly endpoints which coincide.
- (3) γ is not homotopic (rel. endpoints) to a point, nor to a boundary segment between two adjacent marked points.

We denote by $\mathcal{A}(S, M)$ the set of arcs for (S, M) . Thus, in the case when (S, M) is a polygon, $\mathcal{A}(S, M)$ is identified with the set of diagonals.

A *sub-tessellation* (or an *arc system*) of (S, M) is a finite subset $\mathcal{P} \subset \mathcal{A}(S, M)$ which can be represented as by a system of pairwise non-intersecting paths. Note that these paths must also be pairwise non-homotopic, since they represent different elements of $\mathcal{A}(S, M)$. A system of paths representing a sub-tessellation \mathcal{P} cuts S into open pieces S_i° . By the standard “bordification” procedure we complete each S_i° to a compact surface

with boundary S'_i and equip it with a marking M'_i formed by points of M lying inside or on the boundary of S_i° . here we use the standard conventions about degenerate cases:

- (1) If a component S_i° approaches an arc γ on both sides, γ contributes two intervals to $\partial S'_i$.
- (2) If, moreover, an endpoint of γ is not fully encircled by S_i° , it contributes two elements to M'_i . See Figure 3.

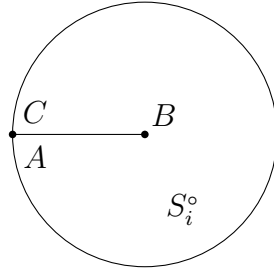


Figure 3: $\partial S'_i$ consists of three segments $[A, B], [B, C], [C, A]$ and $M'_i = \{A, B, C\}$.

Note that each (S'_i, M'_i) is a stable marked surface. By a slight abuse of notation we write $(S'_i, M'_i) \in \mathcal{P}$.

Definition I.1.1. A *tessellation* of (S, M) is a sub-tessellation \mathcal{P} such that each $(S'_i, M'_i) \in \mathcal{P}$ is a polygon. For a tessellation \mathcal{P} we define its Stasheff polytope as the product

$$\mathbb{T}(\mathcal{P}) = \prod_{(S', M') \in \mathcal{P}} \mathbb{T}(S', M').$$

The set $\text{Tess}(S, M)$ of tessellations of (S, M) is partially ordered by inclusion of subsets in $\mathcal{A}(S, M)$. maximal elements are *triangulations* of (S, M) : tessellations for which each (S', M') is a triangle, i.e., $|M'| = 3$, $M' \subset \partial S'$. If \mathcal{P} , then $\mathbb{T}(\mathcal{P}')$ is canonically identified with a face of $\mathbb{T}(\mathcal{P})$.

Definition I.1.2. The *tessellation complex* $\mathbb{T}(S, M)$ is a polyhedral complex obtained by gluing the polytopes $\mathbb{T}(\mathcal{P})$ for all $\mathcal{P} \in \text{Tess}(S, M)$ using the face embeddings $\mathbb{T}(\mathcal{P}') \hookrightarrow \mathbb{T}(\mathcal{P})$ for $\mathcal{P} \leq \mathcal{P}'$.

Note that unlike the above references as well as [23, 26], we do not assume that S is orientable.

Given a tessellation $\mathcal{P} \in \text{Tess}(S, M)$, we associate to it its *dual graph* $\Gamma_{\mathcal{P}}$ by putting one vertex $v_{S', M'}$ inside each polygon $(S', M') \in \mathcal{P}$ and connected the adjacent $v_{(S', M')}$ by edges, similarly to [23, §3.3.3]. When \mathcal{P} varies, the $\Gamma_{\mathcal{P}}$ run over all isotopy classes of spanning graphs for (S, M) . This implies the following.

Proposition I.1.3. $\mathbb{T}(S, M)$ is homotopy equivalent to the nerve of the category $P(S, M)$. \square

We now explain how to prove the following fact which has been widely used in the oriented case, e.g., in [23, 26].

Theorem I.1.4. For any stable marked surface (S, M) (orientable or not) the tessellation complex $\mathbb{T}(S, M)$ is contractible.

Similarly to the oriented case, the proof can be extracted from the results of Harer [33], suitably adjusted, as we now explain.

I.2 The dual cell complex of a triangulated manifold

By a (*regular*) *simplicial complex* we will mean a datum $A = (V, \Sigma)$ where V is a set whose elements are called *vertices* of A , and $\Sigma \subset 2^V$ is a family of finite subsets called *simplices* of A . It is required that:

- All 1-element subsets of V are in Σ .
- If $I \in \Sigma$ and $I' \subset I$, then $I' \in \Sigma$.

The *realization* $|A|$ of a simplicial complex A is obtained in a standard way by gluing the geometric simplices Δ^I associated to $I \in \Sigma$. Explicitly,

$$\Delta^I = \left\{ (p_a)_{a \in I} \in \mathbb{R}^I \mid p_a \geq 0, \sum_{a \in I} p_a = 1 \right\}.$$

A *simplicial subcomplex* of $A = (V, \Sigma)$ is a simplicial complex $A' = (V', \Sigma')$ s.t. $V' \subset V$ and $\Sigma' \subset \Sigma$. In this case $|A'|$ is a closed subset of $|A|$.

By a *space* we will always mean a topological space homeomorphic to $|A| - |A'|$ where A is a simplicial complex and $A' \subset A$ is a subcomplex.

Examples I.2.1. (a) Let X be a space and $\mathcal{U} = (U_a)_{a \in V}$ be a locally finite covering of X by closed subspaces U_a . The *classical nerve* of \mathcal{U} is the simplicial complex $N\mathcal{U} = (V, \Sigma)$ where Σ consists of finite $I \subset V$ s.t. $\bigcap_{v \in I} U_v \neq \emptyset$.

(b) In the situation (a), assume that the covering \mathcal{U} is *saturated*, i.e., for any $U, U' \in \mathcal{U}$ the intersection $U \cap U'$ is either empty or belongs to \mathcal{U} . The *monotone nerve* of \mathcal{U} is the simplicial complex $N^{\leq} \mathcal{U} = (V, \Sigma^{\leq})$, where Σ^{\leq} consists of $I = \{v_0, \dots, v_p\} \subset V$ such that $U_{v_{s(0)}} \subset U_{v_{s(1)}} \subset \dots \subset U_{v_{s(p)}}$ for some permutation s of $\{0, 1, \dots, p\}$. Thus $|N^{\leq}(\mathcal{U})|$ is the same as the realization of the simplicial set which is the nerve of the poset of nonempty $U \in \mathcal{U}$.

The following is classical.

Proposition I.2.2. (a) Assume that any finite intersection $U_{v_0} \cap \dots \cap U_{v_p}$ is either empty or contractible. Then $|N\mathcal{U}|$ is homotopy equivalent to X .

(b) If, moreover, \mathcal{U} is saturated, then $|\mathcal{N}^\leq \mathcal{U}|$ is homotopy equivalent to X . \square

The *barycentric subdivision* of a simplicial complex $A = (V, \Sigma)$ is the simplicial complex $\text{Bar}(A) = (V_{\text{Bar}}, \Sigma_{\text{Bar}})$, where $V_{\text{Bar}} = \Sigma$ and Σ_{Bar} consists of finite subsets $\{I_0, \dots, I_p\}$ such that $I_{s(0)} \subset I_{s(1)} \subset \dots \subset I_{s(p)}$ for some permutation s of $\{0, 1, \dots, p\}$. As well known, $|\text{Bar}(A)|$ is canonically homeomorphic to $|A|$. Notationally, it will be convenient for us to think that vertices of $\text{Bar}(A)$ are formal symbols $\text{bar}(I)$ for $I \in \Sigma$.

For each simplex $I \in \Sigma$ we have the sub complex $A_I \subset \text{Bar}(A)$ called the *link* of I . BY definition, $A_I = (V_I, \Sigma_I)$, where

$$V_I = \{\text{bar}(J) \mid J \supset I, J \in \Sigma\}, \quad \Sigma_I = \{\{\text{bar}(J_0), \dots, \text{bar}(J_p)\}, I \subset J_0 \subset \dots \subset J_p\}.$$

If A' is a subcomplex of A as above, we define the *dual subcomplex* to (A, A') as

$$(A, A')^\vee = \bigcup_{I \in \Sigma - \Sigma'} V_I \subset \text{Bar}(A).$$

Proposition I.2.3. *The realization $|(A, A')^\vee|$ is homotopy equivalent to $|A| - |A'|$.*

Proof: We consider the covering \mathcal{U} of $|A| - |A'|$ by closed subsets $\Delta_0^I = \Delta^I \cap (|A| - |A'|)$ for $I \in \Sigma - \Sigma'$. This covering is saturated, and $(A, A')^\vee = \mathcal{N}^\leq \mathcal{U}$. As each Δ_0^I is contractible, our statement follows from Proposition I.2.2. \square

By a *triangulated manifold* we mean a pair (A, A') consisting of a simplicial complex A and a subcomplex A' such that $|A| - |A'|$ is a topological manifold. In this case each $|V_I|$, $I \in \Sigma - \Sigma'$, is a topological cell which we call the *dual cell* to I . Thus $|(A, A')^\vee|$ is equipped with a cell decomposition into the $|V_I|$.

I.3 Harer's triangulation of the Teichmüller space

Let (S, M) be a stable marked surface. The *arc complex* $A = A(S, M)$ is the simplicial complex with the set of vertices $\mathcal{A}(S, M)$ and simplices being sub-tessellations. It has a subcomplex $A_\infty = A_\infty(S, M)$ formed by sub-tessellations which are not tessellations.

Proposition I.3.1. *The tessellation complex $\mathbb{T}(S, M)$ is homeomorphic to $|(A, A_\infty)^\vee|$ so that each cell $\mathbb{T}(\mathcal{P})$ is identified with $A_{\mathcal{P}}$, the link of \mathcal{P} .*

Proof: Clear by comparing the definitions. \square

The following result was stated and proved by Harer [33] in the oriented case.

Theorem I.3.2. *There is a homeomorphism $\Psi = \Psi_{S, M} : |A| - |A_\infty| \rightarrow \text{Teich}(S, M)$, equivariant with respect to the mapping class group $\text{Mod}(S, M)$.*

Proof: The construction and the argument of [33] extend to the general case as follows. A point $p \in |A| - |A_\infty|$ lies, by definition, strictly inside a simplex $\Delta^{\mathcal{P}}$ for some tessellation \mathcal{P} and thus is represented by the collection of its barycentric coordinates

$$(p_\gamma)_{\gamma \in \mathcal{P}}, \quad p_\gamma > 0, \quad \sum p_\gamma = 1,$$

one for each arc $\gamma \in \mathcal{P}$.

Look at each polygon (S'_i, M'_i) of \mathcal{P} . To each edge e of S'_i we associate the “length” $p_e := p_\gamma$ where γ is the arc of \mathcal{P} that gave rise to e . Let $l_e = \sum_{e \subset \partial S'_i} p_e$ be the total “length” of $\partial S'_i$.

Let S''_i be a disk in \mathbb{C} with center 0 and circumference l_e . We can map M'_i bijectively to a subset $M''_i \subset \partial S''_i$ so that the distances (on $\partial S''_i$) between neighboring elements of M''_i match the corresponding numbers p_e for the edges of S'_i . In this way we get a stable marked Klein surface (S''_i, M''_i) . We now glue the (S''_i, M''_i) together in the same fashion as the (S'_i, M'_i) are glued together to form (S, M) . For this, we use the identifications of the boundary arcs of different S''_i which are affine linear in the standard angle (or arc length) coordinates on the circles $\partial S''_i$. In this way we get a Klein surface (S'', M'') , identified with (S, M) by a diffeomorphism unique up to isotopy. In other words, we get a point of $\text{Teich}(S, M)$. This defines the map $\Psi_{S, M}$. Its $\text{Mod}(S, M)$ -equivariance is clear.

The fact that $\Psi_{S, M}$ is a homeomorphism, was proved in [33] in the oriented case. Suppose S is non-orientable, and let $\varpi : (\tilde{S}, \tilde{M}) \rightarrow (S, M)$ be its orientation cover, with the deck involution τ preserving \tilde{M} . By equivariance, the homeomorphism

$$\Psi_{\tilde{S}, \tilde{M}} : |A(\tilde{S}, \tilde{M})| - |A_\infty(\tilde{S}, \tilde{M})| \longrightarrow \text{Teich}(\tilde{S}, \tilde{M})$$

sends the involution $\tau_A : |A(\tilde{S}, \tilde{M})| \rightarrow |A(\tilde{S}, \tilde{M})|$ induced by τ , to τ_{Teich} . Because of Proposition 3.2.7(b2), we are reduced to the following.

Lemma I.3.3. *We have*

$$(|A(\tilde{S}, \tilde{M})| - |A_\infty(\tilde{S}, \tilde{M})|)^{\tau_A} = |A(S, M)| - |A_\infty(S, M)|.$$

Proof of the lemma: A point $p \in |A(\tilde{S}, \tilde{M})|^{\tau_A}$ lies in a τ -invariant simplex $\Delta^{\tilde{\mathcal{P}}}$ corresponding to a τ -invariant sub-tessellation $\tilde{\mathcal{P}}$ of (\tilde{S}, \tilde{M}) . Let \mathcal{P} be the set of arcs $\varpi(\tilde{\gamma})$, $\tilde{\gamma} \in \tilde{\mathcal{P}}$. We claim that \mathcal{P} is a sub-tessellation for (S, M) . Indeed, the fact that $\tilde{\gamma}$ does not meet $\tau(\tilde{\gamma})$ (which both belong to $\tilde{\mathcal{P}}$) means that $\varpi(\tilde{\gamma})$ does not intersect itself. Further, a homotopy between $\varpi(\tilde{\gamma}_1)$ and $\varpi(\tilde{\gamma}_2)$ implies that $\tilde{\gamma}_2$ is homotopic to either $\tilde{\gamma}_1$ or to $\tau(\tilde{\gamma}_1)$ (because ϖ is an unramified covering). So \mathcal{P} is a sub-tessellation.

Further, suppose that $\tilde{\mathcal{P}}$ is a tessellation. Then any polygon $(\tilde{S}'_i, \tilde{M}'_i)$ of $\tilde{\mathcal{P}}$ must be an unramified covering of its image which therefore must also be a polygon. Therefore \mathcal{P} is a tessellation.

We conclude that

$$p \in (|A(\tilde{S}, \tilde{M})| - |A_\infty(\tilde{S}, \tilde{M})|)^{\tau_A}$$

must lie in a simplex $\Delta^{\tilde{\mathcal{P}}}$ where $\tilde{\mathcal{P}} = \varpi^{-1}(\mathcal{P})$ is the preimage of a tessellation \mathcal{P} of (S, M) and be given by a collection of barycentric coordinates $(p_{\tilde{\gamma}})_{\tilde{\gamma} \in \tilde{\mathcal{P}}}$ which is τ -invariant. Such collections of coordinates are in bijection (homeomorphism) with collections of barycentric coordinates $(p_\gamma)_{\gamma \in \mathcal{P}}$, $\sum p_\gamma = 1$, i.e., with points of the simplex $\Delta^{\mathcal{P}}$ in $|A(S, M)| - |A_\infty(S, M)|$. This proves the lemma and Theorem I.3.2.

Now, Propositions I.1.3, I.3.1 and I.2.3 together with the fact that $\text{Teich}(S, M)$ is homeomorphic to a Euclidean space (Theorem 3.2.7(c)) imply Theorem 4.2.5.

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